Deborah Hugjes-Hallett, Andrew M. Gleason, et al., Calculus, John Wiley & sons, 1994, pages 2-38.

AP Calculus AB and BC – Summer Assignment Name_______________
Chapter 1 – A Library of Functions Pd _____ Date__________

Please complete the following assignments sometime before you return to school in the fall. It would probably be best if you completed them right before the semester begins. We will spend a couple days reviewing this material and then take a quiz on it during the first week. Hopefully, most of this is review for you. This assignment will be collected on the first day of class. Good luck and see you next semester!

Section 1.1 What's A Function?

_____1. Read pp. 2-5. Define the following terms: function, domain, range, mathematical model, directly proportional, inversely proportional.

_____2. Complete problems, pp. 5-7, # 1, 3, 5, 6, 8-10, 12-14.

Section 1.2 Linear Functions

_____3. Read pp. 8-12. Define: linear function, slope, increasing function, decreasing function, extrapolation, interpolation, discrete continuous, implicitly defined function, explicitly defined function, parameters.

_____4. Complete problems, pp. 12-15, # 1-4, 6, 7, 9, 12, 13, 17, 18, 22.

Section 1.3 Exponential Functions

_____5. Read pp. 15-23. Define: concave up, concave down, basic form of an exponential function (Describe what the parameters represent).

_____6. Complete problems, pp. 23-26, # 1-5, 7, 8, 10, 13, 14, 17, 18, 22.

Section 1.4 Power Functions

_____7. Read pp. 27-33.

_____8. Complete problems, pp. 33-35, # 1-3, 6, 7, 13, 15, 19.

Section 1.5 Inverse Functions

_____9. Read pp. 35 – 38.

_____10. Complete problems, pp. 39-40, # 1-16.

1.1 WHAT'S A FUNCTION?

Let's look at an example. In the summer of 1990, the temperatures in Arizona reached an all-time high (so high, in fact, that some airlines decided it might be unsafe to land their planes there). The daily high temperatures in Phoenix for June 19–29 are given in Table 1.1.

TABLE 1.1 *Temperature in Phoenix, Arizona, June 1990*

Date: June (1990)	19	20	21	22	23	24	25	26	27	28	29
Temperature (°F)	109	113	114	113	113	113	120	122	118	118	108

Although you may not have thought of something so unpredictable as temperature as being a function, the temperature *is* a function of date, because each day gives rise to one and only one high temperature. There is no formula for temperature (otherwise we would not need the weather bureau), but nevertheless the temperature does satisfy the definition of a function: Each date, t, has a unique high temperature, H, associated with it.

We define a function as follows:

> One quantity, H, is a **function** of another, t, if each value of t has a unique value of H associated with it. We say H is the *value* of the function or the *dependent variable*, and t is the *argument* or *independent variable*. Alternatively, think of t as the *input* and H as the *output*. We write $H = f(t)$, where f is the name of the function.
> The **domain** of a function is a set of possible values of the independent variable, and the **range** is the corresponding set of values of the dependent variable.

In the temperature example above, the independent variable is the date, and the dependent variable is the temperature. The domain is all possible dates, and the range is the high temperatures on those dates. The function assigns temperatures to dates.

Functions play an important role in science. Frequently, one observes that one quantity is a function of another and then tries to find a reasonable formula to express this function. For example, before about 1590 there was no quantitative idea of temperature. Of course, people understood relative notions like warmer and cooler, and some absolute notions like boiling hot, freezing cold, or body temperature, but there was no numerical measure of temperature. It took the genius of Galileo to realize that the expansion of fluids as they warmed was the key to the measurement of temperature. He was the first to think of temperature as a function of fluid volume.

Finding a function which represents a given situation is called making a *mathematical model.* Such a model can throw light on the relationship between the variables and can thereby help us make predictions.

Representation of Functions: Tables, Graphs, and Formulas

Functions can be represented in at least three different ways: by tables, by graphs, and by formulas. For example, the function giving the temperatures in Phoenix, Arizona, as a function of time can be represented by the graphs in Figure 1.1 as well as by a table.

Other functions arise naturally as graphs. Figure 1.2 contains electrocardiogram (EKG) pictures showing the heartbeat patterns of two patients, one normal and one not. Although it is possible to

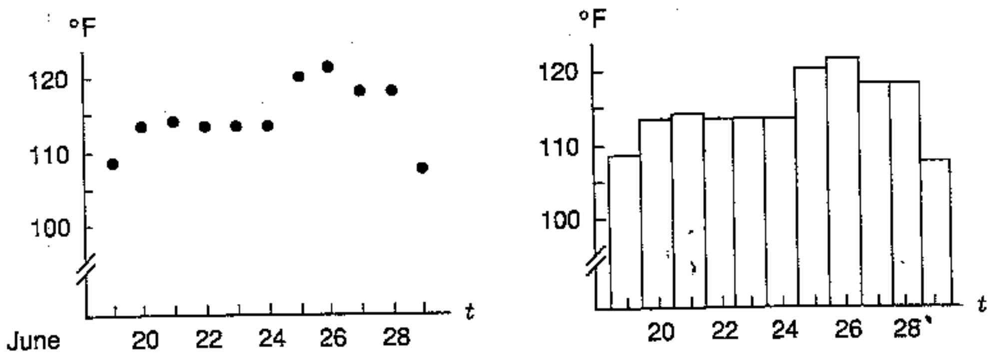


Figure 1.1: Phoenix temperatures, June 1990

construct a formula to approximate an EKG function, this is seldom done. The pattern of repetitions is what a doctor needs to know, and these are much more easily seen from a graph than from a formula. However, each EKG represents a function showing electrical activity as a function of time.

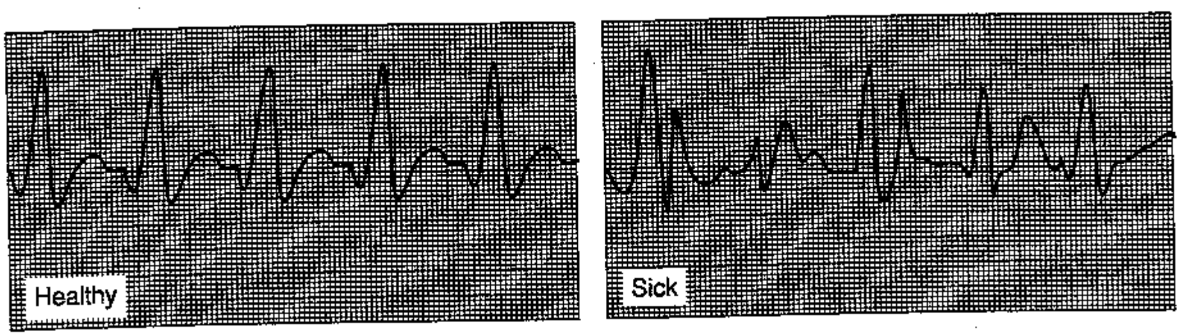


Figure 1.2: EKG readings on two patients

As another example of a function, consider the snow tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine the chirp rate. Even more surprisingly, the chirp rate, C, in chirps per minute, increases steadily with the temperature, T, in degrees Fahrenheit, and to a high degree of accuracy can be computed by the formula

$$C = 4T - 160.$$

The formula for C is written $C = f(T)$ to express the fact that we are thinking of C as a function of T. The graph of this function is in Figure 1.3.

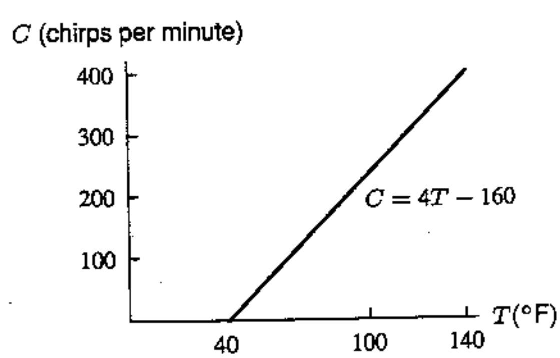


Figure 1.3: Cricket chirp rate versus temperature

Examples of Domain and Range

If the domain of a function is not specified, we will usually take it to be the largest possible set of real numbers. For example, we usually think of the domain of the function $f(x) = x^2$ as all real numbers, whereas the domain of the function $g(x) = 1/x$ is all real numbers except zero. Sometimes, however, we may specify, or restrict, the domain. For example, if the function $f(x) = x^2$ is used to represent the area of a square of side x, we consider only nonnegative values of x and restrict the domain to nonnegative numbers.

Example 1 Consider the function $C = f(T)$ giving chirp rate as a function of temperature. We assume that this equation holds for all temperatures for which the predicted chirp rate is positive, and up to the highest temperature ever recorded at a weather station, namely, 136°F. What is the domain of this function f?

Solution If we consider the equation

$$C = 4T - 160$$

simply as a mathematical relationship between C and T, any T value is possible. However, if we're thinking of it as a relationship between cricket chirps and temperature, then T cannot be less than 40°F, where C falls below the axis and becomes negative. (See Figure 1.3.) In addition, we are told that the formula doesn't hold at temperatures above 136°. Thus, for the function $C = f(T)$ we have

$$\begin{aligned} \text{Domain} &= \text{All } T \text{ values between } 40^\circ\text{F and } 136^\circ\text{F} \\ &= \text{All } T \text{ values with } 40 \le T \le 136. \end{aligned}$$

Therefore we say that the function $C = f(T)$ is represented by the formula

$$C = f(T) = 4T - 160 \quad \text{on the domain} \quad 40 \le T \le 136.$$

Example 2 Find the range of the function f, given the domain from Example 1. In other words, find all possible values of the chirp rate, C, in the equation $C = f(T)$.

Solution Again, if we consider $C = f(T)$ simply as a mathematical relationship, its range is all real C values. However, thinking of its meaning for crickets, the function will predict cricket chirps per minute between 0 (when $T = 40$°F) and 384 (when $T = 136$°F). Hence,

$$\begin{aligned} \text{Range} &= \text{All } C \text{ values from 0 to 384} \\ &= \text{All } C \text{ values with } 0 \le C \le 384. \end{aligned}$$

So far we have used the temperature to predict the chirp rate and thought of the temperature as the *independent variable* and the chirp rate as the *dependent variable*. However, we could do this backwards, and calculate the temperature from the chirp rate. From this point of view, the temperature is dependent on the chirp rate. Thus, which variable is dependent and which is independent may depend on your viewpoint.

Thinking of temperature as a function of chirp rate would enable us (in theory, at least) to use the chirp rate instead of a thermometer to measure temperature. The way we actually do measure temperature is based on another function: the relation between the height of the liquid in a thermometer and temperature. The height of the mercury is certainly a function of temperature; however, we always use this the other way around, and determine the temperature from the height of the mercury, as suggested by Galileo.

Proportionality

A common functional relationship occurs when one quantity is *proportional* to another. For example, if apples are 60 cents a pound, we say the price you pay, p cents, is proportional to the weight you buy, w pounds, because

$$p = f(w) = 60w.$$

As another example, the area, A, of a circle is proportional to the square of the radius, r:

$$A = f(r) = \pi r^2.$$

> In general, y is (directly) **proportional** to x if there is a constant k such that
>
> $$y = kx.$$

We also say that one quantity is *inversely proportional* to another if one is proportional to the reciprocal of the other. For example, the speed, v, at which you make a 50-mile trip is inversely proportional to the time, t, taken, because v is proportional to $1/t$:

$$v = \frac{50}{t} = 50\left(\frac{1}{t}\right).$$

Notice that if y is directly proportional to x, then the magnitude of one variable increases (decreases) when the magnitude of the other increases (decreases). If, however, y is inversely proportional to x, then the magnitude of one variable increases when the value of the other decreases.

Problems for Section 1.1

1. Which of the graphs in Figure 1.4 best match the following three stories?[1] Write a story for the remaining graph.
   (a) I had just left home when I realized I had forgotten my books, and so I went back to pick them up.
   (b) Things went fine until I had a flat tire.
   (c) I started out calmly but sped up when I realized I was going to be late.

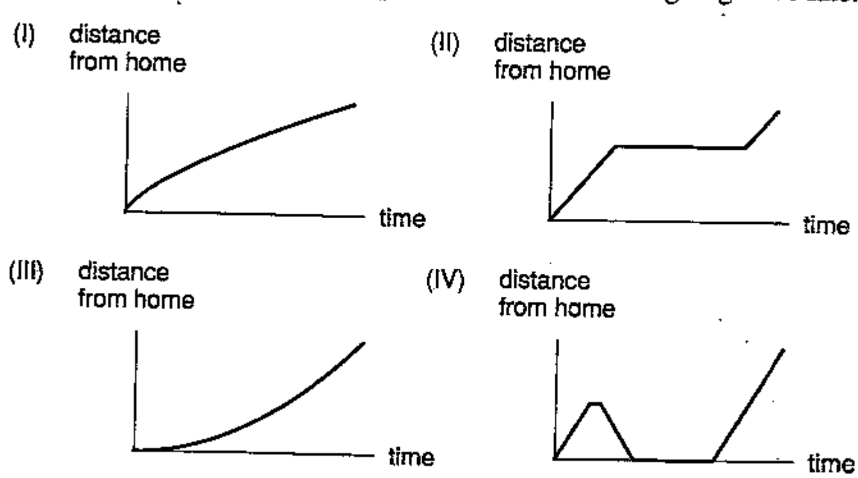


Figure 1.4

[1] Adapted from Jan Terwel. "Real Maths in Cooperative Groups in Secondary Education." In *Cooperative Learning in Mathematics*, edited by Neal Davidson, p 234. (Reading: Addison Wesley, 1990).

2. It warmed up throughout the morning, and then suddenly got much cooler around noon, when a storm came through. After the storm, it warmed up before cooling off at sunset. Sketch a possible graph of this day's temperature as a function of time.
3. Right after a certain drug is administered to a patient with a rapid heart rate, the heart rate plunges dramatically and then slowly rises again as the drug wears off. Sketch a possible graph of the heart rate against time from the moment the drug is administered.
4. Generally, the more fertilizer that is used, the better the yield of the crop. However, if too much fertilizer is applied, the crops become poisoned, and the yield goes down rapidly. Sketch a possible graph showing the yield of the crop as a function of the amount of fertilizer applied.
5. Describe what Figure 1.5 tells you about an assembly line whose productivity is represented as a function of the number of workers on the line.

Figure 1.5

6. A flight from Dulles Airport in Washington, D.C., to LaGuardia Airport in New York City has to circle LaGuardia several times before being allowed to land. Plot a graph of distance of the plane from Washington against time, from the moment of takeoff until landing.
7. In her *Guide to Excruciatingly Correct Behavior*, Miss Manners states:

   > There are three possible parts to a date of which at least two must be offered: entertainment, food and affection. It is customary to begin a series of dates with a great deal of entertainment, a moderate amount of food and the merest suggestion of affection. As the amount of affection increases, the entertainment can be reduced proportionately. When the affection has replaced the entertainment, we no longer call it dating. Under no circumstances can the food be omitted.

   Based on this statement, sketch a graph showing entertainment as a function of affection, assuming the amount of food to be constant. Mark the point on the graph at which the relationship starts, as well as the point at which the relationship ceases to be called dating.

Problems 8 and 9 are about supply and demand curves. Economists are interested in how the quantity of an item which is manufactured and sold, q, depends on its price, p. They think of quantity as a function of price. However, for historical reasons[2] the economists put price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis. Since manufacturers and consumers react differently to changes in price, there are two functions relating p and q. The *supply curve* represents how the quantity of an item that manufacturers are willing to supply depends on the price for which the item can be sold. The *demand curve* represents how the quantity of an item demanded by consumers depends on its price.

[2]Originally, the economists thought of price as the dependent variable and put it on the vertical axis. Unfortunately, when their point of view changed, the axes did not.

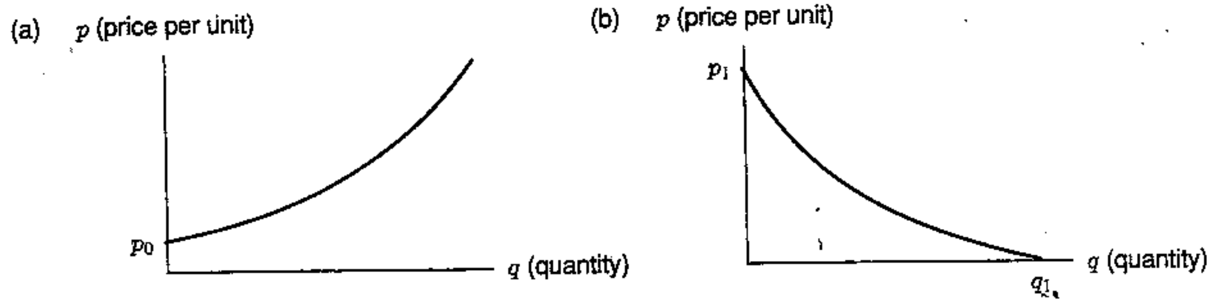


Figure 1.6

8. One of the graphs in Figure 1.6 is a supply curve, and the other is a demand curve. Which is which? Why?
9. The price p_0 in Figure 1.6(a) represents the price below which the manufacturers are unwilling to produce any of the item. What do the price p_1 and the quantity q_1 in Figure 1.6(b) represent in practical economic terms?
10. Specify the domain and range of the function $y = f(x)$ whose graph is shown in Figure 1.7.

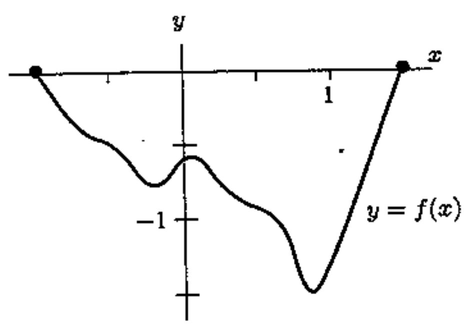


Figure 1.7

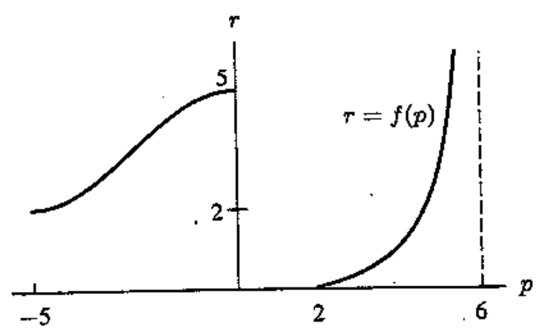


Figure 1.8

11. The graph of $r = f(p)$ is given in Figure 1.8.
    (a) What could be the domain of f?
    (b) What could be the range of f?
    (c) What values of r could correspond to exactly one value of p?
12. If $g(y) = 1/(y^2 - y)$, find all y values which do not determine a real value for $g(y)$. Solve $g(y) = 1/2$.
13. If $y = f(x) = 1/\sqrt{4 - x^2}$, what values of x do not determine a real value for y? Solve $f(x) = 5$.
14. When Galileo was formulating the laws of motion, he considered the motion of a body starting from rest and falling under gravity. He originally thought that the velocity of such a falling body was proportional to the distance it had fallen. What light does the data in Table 1.2 shed on Galileo's hypothesis? What alternative hypothesis is suggested by the two sets of data in Table 1.2 and Table 1.3?

TABLE 1.2

Distance (ft)	0	1	2	3	4
Velocity (ft/sec)	0	8	11.3	13.9	16

TABLE 1.3

Time (sec)	0	1	2	3	4
Velocity (ft/sec)	0	32	64	96	128

1.2 LINEAR FUNCTIONS

Probably the most commonly used functions are the *linear functions*. These are functions that represent a steady increase or a steady decrease. A function is linear if any change, or increment, in the independent variable causes a proportional change, or increment, in the dependent variable.

The Olympic Pole Vault

During the early years of the Olympics, the height of the winning pole vault increased approximately as shown in Table 1.4. Since the winning height increased regularly by 8 inches every four years, the height is a linear function of time over the period from 1900 to 1912. The height starts at 130 inches and increases by the equivalent of 2 inches every year, so if y is the height in inches and t is the number of years since 1900, we can write

$$y = f(t) = 130 + 2t.$$

The coefficient 2 tells us the rate at which the height increases and is the *slope* of the line $f(t) = 130 + 2t$.

TABLE 1.4 *Olympic pole vault records (approximate)*

Year	1900	1904	1908	1912
Height (inches)	130	138	146	154

You can visualize the slope in Figure 1.9 as the ratio

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{8}{4} = 2.$$

Calculating the slope (rise/run) using any other two points on the line gives the same value. It is this fact—that the slope, or rate of change, is the same everywhere—that makes a line straight. For a function that is not linear, the rate of change will vary from point to point. Since $y = f(t)$ increases with t, we say that f is *an increasing function.* What about the constant 130? This represents the initial height in 1900, when $t = 0$. Geometrically, the 130 is the *intercept* on the vertical axis.

You may wonder whether the linear trend continues beyond 1912. Not surprisingly, it doesn't exactly. The formula $y = 130 + 2t$ predicts that the height in the 1988 Olympics would be 306 inches or 25 feet 6 inches, which is considerably higher than the actual value of 19 feet 9 inches. In fact, the height does increase at almost every session of the Olympics, but not at a constant rate. Thus, there is clearly a danger in *extrapolating* too far from the given data. You should also observe that the data in Table 1.4 is *discrete*, because it is given only at specific points (every four years). However, we have treated the variable t as though it were *continuous*, because the function $y = 130 + 2t$ makes sense for all values of t. The graph in Figure 1.9 is of the continuous function because it is a solid line, rather than four separate points representing the years in which the Olympics were held.

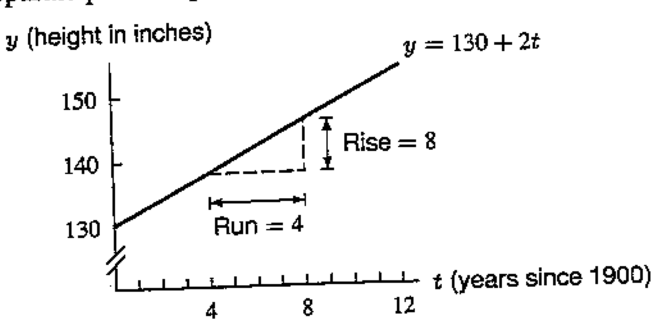


Figure 1.9: Olympic pole vault records

Linear Functions in General

> **A linear function** has the form
>
> $$y = f(x) = b + mx.$$
>
> Its graph is a line such that
> - m is the slope, or rate of change of y with respect to x.
> - b is the vertical intercept, or value of y when x is zero.

Notice that if the slope is zero, $m = 0$, we have $y = b$, a horizontal line.

> To recognize that a function $y = f(x)$ given by a table of data is linear, look for differences in y values that are constant for equal differences in x.

The slope of a linear function can be calculated from values of the function at two points, given by a and c, using the formula

$$m = \frac{\text{Rise}}{\text{Run}} = \frac{f(c) - f(a)}{c - a}.$$

The quantity $(f(c) - f(a))/(c - a)$ is called a *difference quotient* because it is the quotient of two differences. (See Figure 1.10). In Chapter 2, you will see that difference quotients play an important role in calculus.

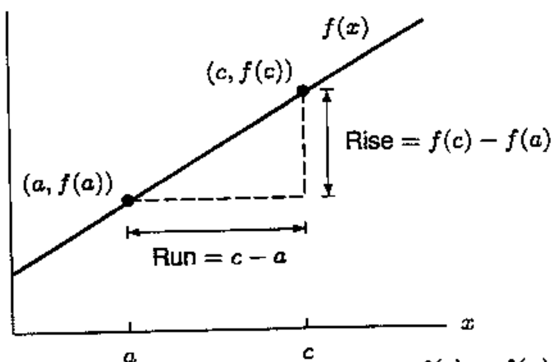


Figure 1.10: Difference quotient $= \dfrac{f(c) - f(a)}{c - a}$

The Success of Search and Rescue Teams

Consider the problem of the "search and rescue" teams working to find lost hikers in remote areas in the West. To search for an individual, members of the search team separate and walk parallel to one another through the area to be searched. Experience has shown that the team's chance of finding a lost individual is related to the distance, d, by which team members are separated. The percentage found[3] for various separations are recorded in Table 1.5.

[3]From *An Experimental Analysis of Grid Sweep Searching*, by J. Wartes (Explorer Search and Rescue, Western Region, 1974).

TABLE 1.5 *Separation of searchers versus success rate*

Separation distance d (ft)	Percent found, P
20	90
40	80
60	70
80	60
100	50

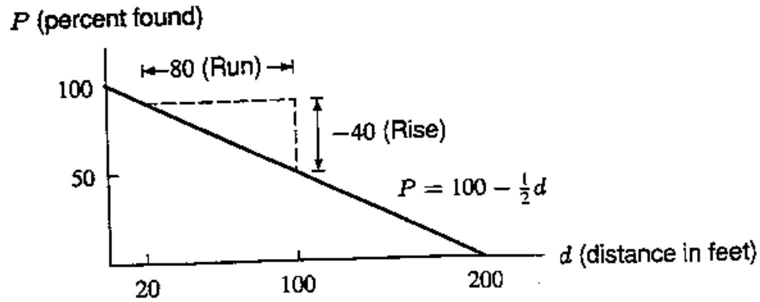


Figure 1.11: Separation of searchers versus success rate

From the data in the table, you can see that as the separation distance decreases, a larger percentage of the lost hikers is found, which makes sense. Since $P = f(d)$ decreases as d increases, we say that P is a *decreasing function* of d. You can also see that for the data given, each 20-foot increase in distance causes the percentage found to drop by 10. This constant decrease in P as d increases by a fixed amount is a clear indication that the graph of P against d is a line. (See Figure 1.11.) Notice that the slope is $-40/80 = -1/2$. The negative sign shows that P decreases as d increases. The slope is the rate at which P is increasing or decreasing, as d increases.

What about the vertical intercept? If $d = 0$, the searchers are walking shoulder to shoulder and you'd expect everyone to be found, so $P = 100$. This is exactly what you get if the line is continued to the vertical axis (a decrease of 20 in d causes an increase of 10 in P). Therefore, the equation of the line is

$$P = f(d) = 100 - \frac{1}{2}d.$$

What about the horizontal intercept? When $P = 0$, or $0 = 100 - \frac{1}{2}d$, then $d = 200$. The value $d = 200$ represents the separation distance at which, according to the model, no one is found. This is unreasonable, because even when the searchers are far apart, the search will sometimes be successful. This suggests that somewhere outside the given data, the linear relationship ceases to hold. As in the pole vault example, extrapolating too far beyond the given data may not give accurate answers.

Increasing versus Decreasing Functions

Let's summarize what we know about increasing and decreasing functions:

> A function f is **increasing** if the values of $y = f(x)$ increase as x increases.
> A function f is **decreasing** if the values of $y = f(x)$ decrease as x increases.
>
> The graph of an *increasing* function *climbs* as you move from left to right.
> The graph of a *decreasing* function *descends* as you move from left to right.

A Budget Constraint

An ongoing debate in the federal government concerns the allocation of money between defen and social programs. In general, the more that is spent on defense, the less that is available f social programs, and vice versa. Let's simplify the example to guns and butter. Assuming a constant budget, we will show that the relationship between the number of guns and the quantity of butter is linear. Suppose there is \$12,000 to be spent and that it is to be divided between guns, costing \$400 each, and butter, costing \$2000 a ton. Suppose the number of guns bought is g, and the number of tons of butter is b. Then the amount of money spent on guns is \$400g (because each one is \$400), and the amount spent on butter is \$2000b. Assuming all the money is spent,

$$\begin{array}{c}\text{Amount spent}\\ \text{on guns}\end{array} + \begin{array}{c}\text{Amount spent}\\ \text{on butter}\end{array} = \$12{,}000$$

or

$$400g + 2000b = 12{,}000$$

or

$$g + 5b = 30.$$

This equation is the budget constraint. Its graph is the line shown in Figure 1.12, which can be found by plotting points. We will calculate the points at which the graph crosses the axes. If $b = 0$, then

$$g + 5(0) = 30 \quad \text{so} \quad g = 30.$$

If $g = 0$, then

$$0 + 5b = 30 \quad \text{so} \quad b = 6.$$

Since the number of guns bought determines the amount of butter bought (because all the money which doesn't go to guns goes to butter), b is a function of g. Similarly, the amount of butter bought determines the number of guns, so g is a function of b. The budget constraint represents an *implicitly defined function*, because neither quantity is given explicitly in terms of the other. If we solve for g, giving

$$g = 30 - 5b$$

we have an *explicit* formula for g in terms of b. Similarly,

$$b = \frac{30 - g}{5} \quad \text{or} \quad b = 6 - 0.2g$$

gives b as an explicit function of g. Since the explicit functions

$$g = 30 - 5b \quad \text{and} \quad b = 6 - 0.2g$$

are linear, the graph of the budget constraint must be a line.

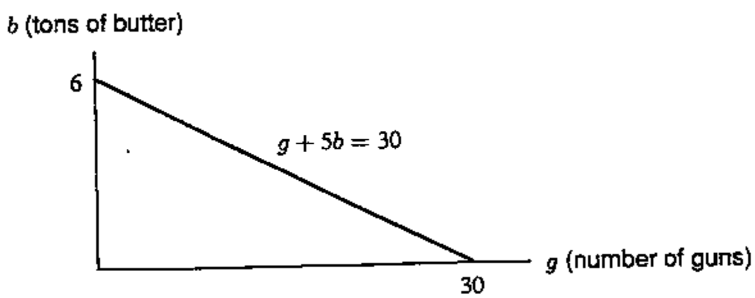


Figure 1.12: Budget constraint

Families of Linear Functions

Formulas such as $f(x) = mx$, and $f(x) = b + mx$, containing constants such as m and b which can take on various values, are said to define a *family of functions*. The constants m and b are called *parameters*. Each of the functions in this section belongs to the family $f(x) = b + mx$.

Grouping functions into families which share important features is particularly useful for mathematical modeling. We often choose a family to represent a given situation on theoretical grounds and then use data to determine the particular values of the parameters. The meaning of the parameters m and b in the family $f(x) = b + mx$ is shown in Figures 1.13 and 1.14.

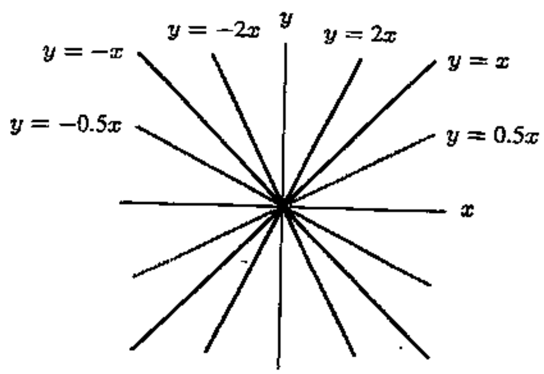


Figure 1.13: The family $y = mx$ (with $b = 0$)

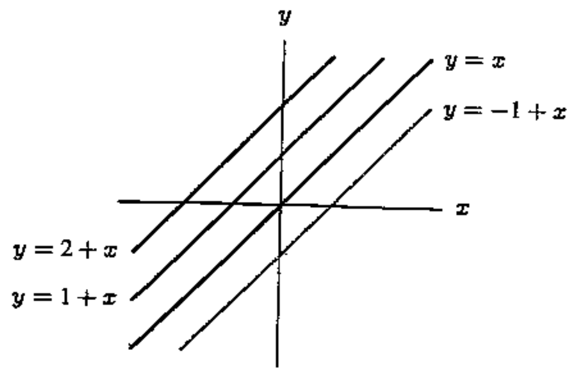


Figure 1.14: The family $y = b + x$ (with $m = 1$)

Problems for Section 1.2

1. Find the slope and vertical intercept of the line whose equation is $2y + 5x - 8 = 0$.
2. Find the equation of the line through the points $(-1, 0)$ and $(2, 6)$.
3. Find the equation of the line with slope m through the point (a, c).

For Problems 4–5, recall that parallel lines have equal slopes, and that two lines are perpendicular if their slopes are negative reciprocals.

4. Find the equation of the line through the point $(2, 1)$ which is perpendicular to the line $y = 5x - 3$.
5. Find the equations of the lines parallel to and perpendicular to the line $y + 4x = 7$ through the point $(1, 5)$.
6. Estimate the slope of the line shown in Figure 1.15 and use the slope to find an equation for that line. (Note that the x and y scales are unequal.)

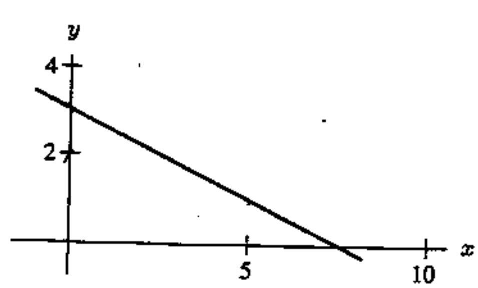


Figure 1.15

7. Match the graphs in Figure 1.16 with the equations below.

(a) $y = x - 5$
(b) $-3x + 4 = y$
(c) $5 = y$
(d) $y = -4x - 5$
(e) $y = x + 6$
(f) $y = x/2$

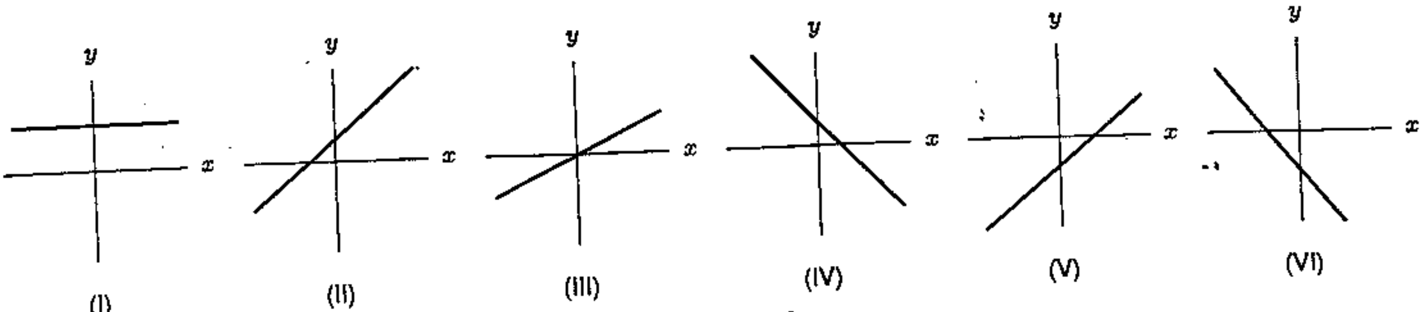


Figure 1.16

8. Match the graphs in Figure 1.17 with the equations below.

(a) $y = -2.72x$
(b) $y = 0.01 + 0.001x$
(c) $y = 27.9 - 0.1x$
(d) $y = 0.1x - 27.9$
(e) $y = -5.7 - 200x$
(f) $y = x/3.14$

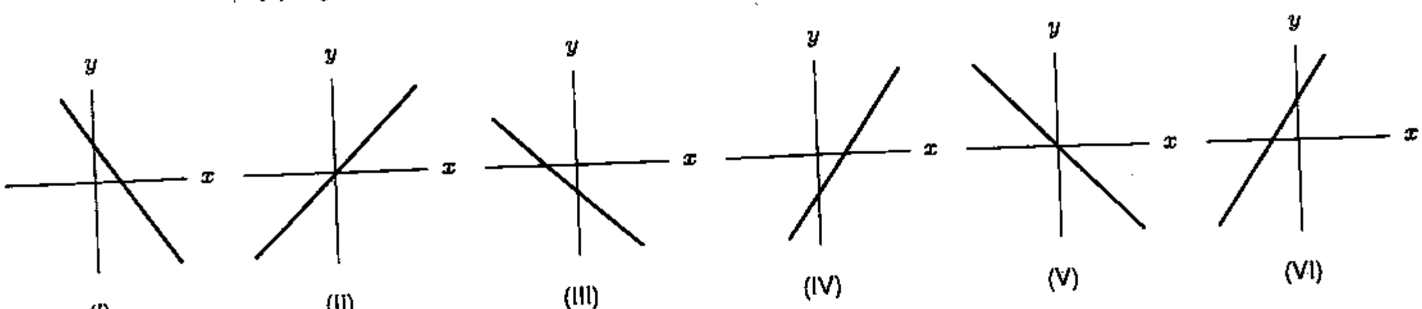


Figure 1.17

9. Corresponding values of p and q are given in the table below.

(a) Find q as a linear function of p.
(b) Find p as a linear function of q.

p	1	2	3	4
q	950	900	850	800

10. A linear equation was used to generate the values in the table below. Find that equation.

x	5.2	5.3	5.4	5.5	5.6
y	27.8	29.2	30.6	32.0	33.4

11. An equation of a line is $3x + 4y = -12$. Find the length of the portion of the line that lies between its x and y intercepts.

12. A car rental company offers cars at \$40 a day and 15 cents a mile. Its competitor's cars are \$50 a day and 10 cents a mile.

(a) For each company, write a formula giving the cost of renting a car for a day as a function of the distance traveled.
(b) On the same axes, sketch graphs of both functions.
(c) How should you decide which company is cheaper?

13. Consider a graph of Fahrenheit temperature, °F, against Celsius temperature, °C, and assume that the graph is a line. You know that 212°F and 100°C both represent the temperature at which water boils. Similarly, 32°F and 0°C both represent water's freezing point.
    (a) What is the slope of the graph?
    (b) What is the equation of the line?
    (c) Use the equation to find what Fahrenheit temperature corresponds to 20°C.
    (d) What temperature is the same number of degrees in both Celsius and Fahrenheit?
14. The cost of planting seed is usually a function of the number of acres sown. The cost of the equipment is a *fixed cost* because it must be paid regardless of the number of acres planted. The cost of supplies and labor varies with the number of acres planted and are called *variable costs*. Suppose the fixed costs are $10,000 and the variable costs are $200 per acre. Let C be the total cost, measured in thousands of dollars, and let x be the number of acres planted.
    (a) Find a formula for C as a function of x.
    (b) Sketch a graph of C against x.
    (c) Explain how you can visualize the fixed and variable costs on the graph.
15. Suppose you are driving at a constant speed from Chicago to Detroit, about 275 miles away. About 120 miles from Chicago you pass through Kalamazoo, Michigan. Sketch a graph of your distance from Kalamazoo as a function of time.
16. Hot peppers have been rated according to Scoville units, with a maximum human tolerance level of 14,000 Scovilles per dish. The West Coast Restaurant, known for spicy dishes, promises a daily special to satisfy the most avid spicy-dish fans. The restaurant imports Indian peppers rated at 1200 Scovilles each and Mexican peppers with a Scoville rating of 900 each.
    (a) Determine the Scoville constraint equation relating the maximum number of Indian and Mexican peppers the restaurant should use for their specialty dish.
    (b) Solve the equation from part (a) to show explicitly the number of Indian peppers needed in the hottest dishes as a function of the number of Mexican peppers.
17. You have a fixed budget of $\$k$ to spend on soda and suntan oil, which cost $\$p_1$ per liter and $\$p_2$ per liter respectively.
    (a) Write an equation expressing the relationship between the number of liters of soda and the number of liters of suntan oil that you can buy if you exhaust your budget. This is your *budget constraint*.
    (b) Graph the budget constraint, assuming that you can buy fractions of a liter. Label the intercepts.
    (c) Suppose your budget is suddenly doubled. Graph the new budget constraint on the same axes.
    (d) With a budget of $\$k$, the price of suntan oil suddenly doubles. Sketch the new budget constraint on the same axes.
18. Since the opening up of the West, the US population has moved westward. To observe this, we look at the "population center" of the US, which is the point at which the country would balance if it were a flat plate with no weight, and every person had equal weight. In 1790 the population center was east of Baltimore, Maryland. It has been moving westward ever since, and in 1990 it crossed the Mississippi river to Steelville, Missouri (southwest of St. Louis). During the second half of this century, the population center has moved about 50 miles west every 10 years.
    (a) Express the approximate position of the population center as a function of time, measured in years from 1990. Measure position westward from Steelville, along the line running

through Baltimore.

(b) The distance from Baltimore to St. Louis is a bit over 700 miles. Could the population center have been moving at roughly the same rate for the last two centuries?

(c) Could the function in part (a) continue to apply for the next three centuries? Why or why not? [Hint: You may want to look at a map. Note that distances are in air miles and are not driving distances.]

19. For small changes in temperature, the formula for the expansion of a metal rod under a change in temperature is:

$$l - l_0 = al_0(t - t_0),$$

where l is the length of the object at temperature t, and l_0 is the initial length at temperature t_0, and a is a constant which depends on the type of metal.

(a) Express l as a linear function of t. Find the slope and y-intercept. [Hint: Treat the other quantities as constants.]

(b) Suppose you had a rod which was initially 100 cm long at 60°F and made of a metal with a equal to 10^{-5}. Write an equation giving the length of this rod at temperature t.

(c) What does the sign of the slope of the graph tell you about the expansion of a metal under a change in temperature?

20. When a cold yam is put into a hot oven to bake, the temperature of the yam rises. The rate, R (in degrees per minute), at which the temperature of the yam rises is governed by Newton's Law of Heating, which says that the rate is proportional to the temperature difference between the yam and the oven. If the oven is at 350°F and the temperature of the yam is H°F:

(a) Write a formula giving R as a function of H.

(b) Sketch the graph of R against H.

21. When a cup of hot coffee sits on the kitchen table, its temperature falls. The rate, R, at which its temperature changes is governed by Newton's Law of Cooling, which says that the rate is proportional to the temperature difference between the coffee and the surrounding air. Let's think of the rate, R, as a negative quantity because the temperature of the coffee is falling. If the temperature of the coffee is H°C and the temperature of the room is 20°C:

(a) Write a formula giving R as a function of H.

(b) Sketch a graph of R against H.

22. A body of mass m is falling downward with velocity v. Newton's Second Law of Motion, $F = ma$, says that the net downward force, F, on the body is proportional to its downward acceleration, a. The net force, F, consists of the force due to gravity, F_g, which acts downward, minus the air resistance, F_r, which acts upward. The force due to gravity is mg, where g is a constant. Assume the air resistance is proportional to the velocity of the body.

(a) Write an expression for the net force, F, as a function of the velocity, v.

(b) Write a formula giving a as a function of v.

(c) Sketch a against v.

1.3 EXPONENTIAL FUNCTIONS

Population Growth

Consider the data for the population of Mexico in the early 1980s in Table 1.6. To see how the population is growing, you might look at the increase in population from one year to the next, as shown in the third column. If the population had been growing linearly, all the numbers in the third column would be the same. But populations usually grow faster as they get bigger, because there are more people to have babies. So you shouldn't be surprised to see the numbers in the third column increasing.

TABLE 1.6 *Population of Mexico (estimated), 1980–1986*

Year	Population (millions)	Change in population (millions)
1980	67.38	
		1.75
1981	69.13	
		1.80
1982	70.93	
		1.84
1983	72.77	
		1.89
1984	74.66	
		1.94
1985	76.60	
		1.99
1986	78.59	

Suppose we divide each year's population by the previous year's population. We get, approximately,

$$\frac{\text{Population in 1981}}{\text{Population in 1980}} = \frac{69.13 \text{ million}}{67.38 \text{ million}} = 1.026$$

$$\frac{\text{Population in 1982}}{\text{Population in 1981}} = \frac{70.93 \text{ million}}{69.13 \text{ million}} = 1.026.$$

The fact that both calculations give 1.026 shows the population grew by about 2.6% between 1980 and 1981 *and* between 1981 and 1982. If you do similar calculations for other years, you will find that the population grew by a factor of about 1.026, or 2.6%, every year. Whenever you have a constant growth factor (here 1.026), you have *exponential growth*. If t is the number of years since 1980,

$$\begin{aligned}
\text{When } t = 0,&\quad \text{population} = 67.38 = 67.38(1.026)^0 \\
\text{When } t = 1,&\quad \text{population} = 69.13 = 67.38(1.026)^1 \\
\text{When } t = 2,&\quad \text{population} = 70.93 = 69.13(1.026) = 67.38(1.026)^2 \\
\text{When } t = 3,&\quad \text{population} = 72.77 = 70.93(1.026) = 67.38(1.026)^3
\end{aligned}$$

and so t years after 1980, the population is given by

$$P = 67.38(1.026)^t.$$

This is an *exponential function* with base 1.026. It is called exponential because the variable, t, is in the exponent. The base represents the factor by which the population grows each year.

If we assume that the same formula will hold for the next 50 years or so, the population will have the shape shown in Figure 1.18. Since the population is growing, the function is increasing. Notice also that the population grows faster and faster as time goes on. This behavior is typical of an exponential function. You should compare this with the behavior of a linear function, which climbs at the same rate everywhere and so has a straight-line graph. Because this graph is bending upward, we say it is *concave up*. Even exponential functions which climb slowly at first, such as this one, climb extremely quickly eventually. That is why exponential population growth is such a threat to the world.

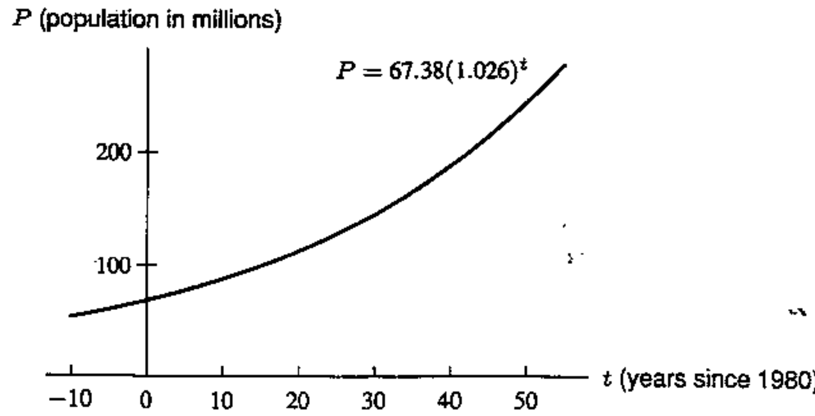


Figure 1.18: Population of Mexico (estimated): Exponential growth

Even if it represents reliable data, the smooth graph in Figure 1.18 is actually only an approximation to the true graph of the population of Mexico. Since we can't have fractions of people, the graph should really be jagged, jumping up or down by one each time someone is born or dies. However, with a population in the millions, the jumps are so small as to be invisible at the scale we are using. Therefore, the smooth graph is an extremely good approximation.

Example 1 Predict the population of Mexico in the year
(a) 2007 (when $t = 27$). (b) 2034 (when $t = 54$). (c) 2061 (when $t = 81$).

Solution Extrapolating so far into the future can be risky because it assumes that the population continues to grow exponentially with the same constant growth factor. (There could, for example, be a medical breakthrough that would increase the growth factor, or an epidemic that would decrease it.) Writing "$\approx$" to represent approximately equal, the model we are using predicts
(a) $P = 67.38(1.026)^{27} \approx 67.38(2) = 134.76$ million.
(b) $P = 67.38(1.026)^{54} \approx 67.38(4) = 269.52$ million.
(c) $P = 67.38(1.026)^{81} \approx 67.38(8) = 539.04$ million.

If you look at the answers to Example 1, you will see something that may surprise you. After 27 years the population has doubled; after another 27 years (at $t = 54$), it has doubled again. In another 27 years later (when $t = 81$), the population has doubled yet again. As a result, we say that the *doubling time* of the population of Mexico is 27 years.

Every exponentially growing population has a fixed doubling time. The world's population currently has a doubling time of about 38 years. Notice what this means: If you live to be 76, the world's population is expected to quadruple in your lifetime.

Concavity

We have used the term concave up to describe the graph in Figure 1.18. In Figure 1.21 on page 21, we will see a graph which is concave down. In general:

> The graph of any function is **concave up** if it bends upward, and it is **concave down** if it bends downward. A line is neither concave up nor concave down.

Musical Pitch

The pitch of a musical note is determined by the frequency of the vibration which causes it. Middle C on the piano, for example, corresponds to a vibration of 263 hertz (cycles per second). A note one octave above middle C vibrates at 526 hertz, and a note two octaves above middle C vibrates at 1052 hertz. (See Table 1.7.)

TABLE 1.7 *Pitch of notes above middle C*

Number, n, of octaves above middle C	Number of hertz $V = f(n)$
0	263
1	526
2	1052
3	2104
4	4208

TABLE 1.8 *Pitch of notes below middle C*

n	$V = 263 \cdot 2^n$
−3	$263 \cdot 2^{-3} = 263(1/2^3) = 32.875$
−2	$263 \cdot 2^{-2} = 263(1/2^2) = 65.75$
−1	$263 \cdot 2^{-1} = 263(1/2) = 131.5$
0	$263 \cdot 2^0 = 263$

Notice that

$$\frac{526}{263} = 2 \quad \text{and} \quad \frac{1052}{526} = 2 \quad \text{and} \quad \frac{2104}{1052} = 2$$

and so on. In other words, each value of V is twice the value before, so

$$f(1) = 526 = 263 \cdot 2 = 263 \cdot 2^1$$
$$f(2) = 1052 = 526 \cdot 2 = 263 \cdot 2^2$$
$$f(3) = 2104 = 1052 \cdot 2 = 263 \cdot 2^3.$$

In general

$$V = f(n) = 263 \cdot 2^n.$$

The base 2 represents the fact that as we go up an octave, the frequency of vibrations doubles. Indeed, our ears hear a note as one octave higher than another precisely because it vibrates twice as fast. For the negative values of n in Table 1.8, this function represents the octaves below middle C. The notes on a piano are represented by values of n between −3 and 4, and the human ear finds values of n between −4 and 7 audible.

Although $V = f(n) = 263 \cdot 2^n$ makes sense in musical terms only for certain values of n, values of the function $f(x) = 263 \cdot 2^x$ can be calculated for all real x, and its graph has the typical exponential shape, as can be seen in Figure 1.19. It is concave up, climbing faster and faster as x increases.

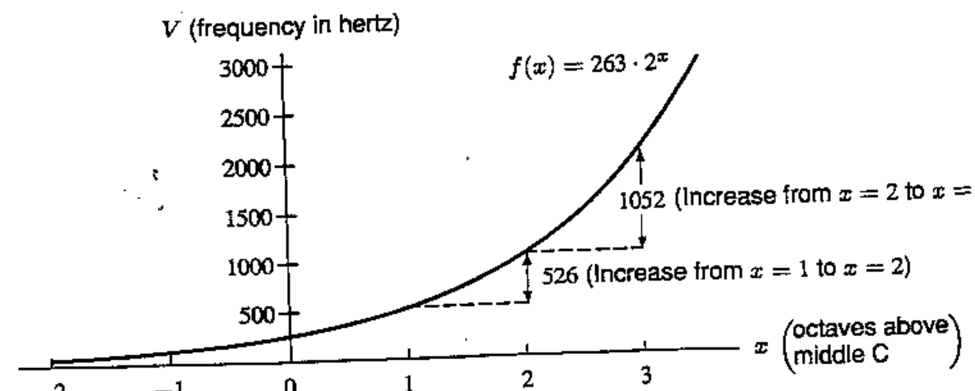


Figure 1.19: Pitch as a function of number of octaves above middle C

Removal of Pollutants from Jet Fuel

Now we will look at an example in which a quantity is decreasing instead of increasing. Before kerosene can be used as jet fuel, federal regulations require that the pollutants in it be removed by passing the kerosene through clay. We will suppose the clay is in a pipe and that each foot of the pipe removes 20% of the pollutants that enter it. Therefore each foot leaves 80% of the pollution. If P_0 is the initial quantity of pollutant and $P = f(n)$ is the quantity left after n feet of pipe:

$$\begin{aligned} f(0) &= P_0 \\ f(1) &= (0.8)P_0 \\ f(2) &= (0.8)(0.8)P_0 = (0.8)^2 P_0 \\ f(3) &= (0.8)(0.8)^2 P_0 = (0.8)^3 P_0 \end{aligned}$$

and so, after n feet,

$$P = f(n) = P_0(0.8)^n.$$

In this example, n must be non-negative. However, the *exponential decay function*

$$P = f(x) = P_0(0.8)^x$$

makes sense for any real x. We'll plot it with $P_0 = 1$ in Figure 1.20; some values of the function are in Table 1.9.

Notice the way the function in Figure 1.20 is decreasing: each downward step is smaller than the one before. This is because as the kerosene gets cleaner, there's less dirt to remove, and so each foot of clay takes out less pollutant than the previous one. Compare this to the exponential growth in Figures 1.18 and 1.19 on pages 17 and 18, where each step upward is larger than the one before. Notice, however, that all three graphs mentioned are concave up.

TABLE 1.9 *Values of decay function*

x	$P = (0.8)^x$
−2	1.56
−1	1.25
0	1
1	0.8
2	0.64
3	0.51
4	0.41

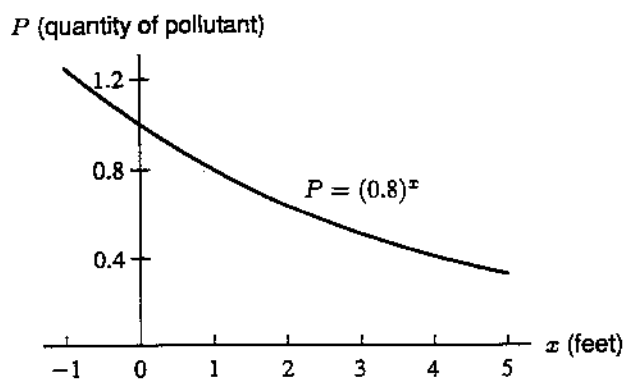


Figure 1.20: Pollutant removal: Exponential decay

The General Exponential Function

> P is an **exponential function** of t with base a if
>
> $$P = P_0 a^t$$
>
> where P_0 is the initial quantity (when $t = 0$) and a is the factor by which P changes when t increases by 1.
> If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.

The largest possible domain for the exponential function is all real numbers, provided $a > 0$. (Why do we not want $a \leq 0$?)

> To recognize that a function $P = f(t)$ given by a table of data is exponential, look for ratios of P values that are constant for equally spaced t values.

Radioactive Decay

Radioactive substances, such as uranium, decay by a certain percentage of their mass in a given unit of time. The most common way to express this rate of decay is to give the time period it takes for half the mass to decay. This period of time is called the *half-life* of the substance. The important thing to remember about radioactive decay is that two half-lives do not make a whole life! Rather, in the time period of two half-lives, the substance will decay to $(1/2) \cdot (1/2) = 1/4$ of its original mass.

One of the most well-known radioactive substances is carbon-14, which is used to date organic objects. When the object, such as a piece of wood or bone, was part of a living organism, it accumulated small amounts of radioactive carbon-14, so that a certain proportion of the carbon in the object was carbon-14. Once the organism dies, it no longer picks up carbon-14 through interaction with its environment (for example, through respiration).

By measuring the proportion of carbon-14 in the object and comparing that to the proportion in living material, we can estimate how much of the original carbon-14 has decayed. The half-life of carbon-14 is about 5730 years. Thus, after roughly 5000 years, we would find the object had about $1/2$ as much carbon-14 as when it was alive. After 10,000 years, we would find about $1/4$ as much, and after 15,000 years, about $1/8$ as much. We can write an exponential function for the amount of carbon-14 left after a period of time t. First, suppose we measure time in units of 5730 years. Then if C_0 was the original amount of carbon-14, the amount, C, of carbon left after T "units" of time (namely, T half-lives), would be $C = C_0(1/2)^T$. However, we usually do not measure time in units of 5730 years, so if we let t be time measured in years (units of one year), then $T = t/5730$, and

$$C = C_0\left(\frac{1}{2}\right)^{(t/5730)}.$$

In general, if a substance has a half-life of h years (or minutes or seconds), then the quantity, Q, of the substance left after t units of time, if there was Q_0 of the substance originally, is

$$Q = Q_0\left(\frac{1}{2}\right)^{(t/h)}.$$

In summary, we use the following definitions:

> The doubling time of an exponentially increasing quantity is the time for it to double.
> The half-life of an exponentially decaying quantity is the time for it to be reduced to half.

Drug Buildup

Suppose that we want to model the amount of a certain drug in the body. Imagine that initially there is none, but that the quantity slowly starts to increase via a continuous intravenous injection. As the quantity in the body increases, so does the rate at which the body excretes the drug, so that eventually the quantity levels off at a saturation value, S. The graph of quantity against time will look something like that in Figure 1.21.

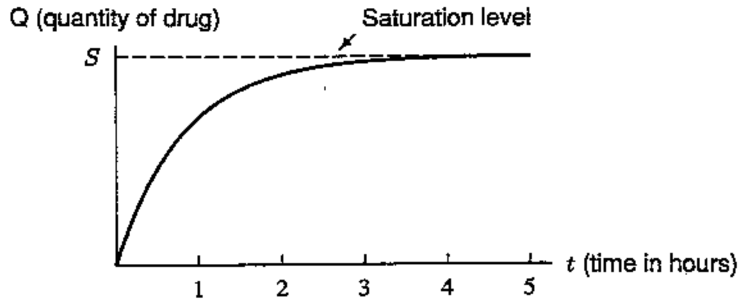


Figure 1.21: Buildup of drug in body

Notice that the quantity, Q, starts at zero and increases toward S. We say that the line representing the saturation level is a *horizontal asymptote*, because the graph gets closer and closer to it as time increases. Since the rate at which the quantity of the drug increases slows as it approaches S, this graph is bending downward; hence it is *concave down.*

Suppose we want to make a mathematical model of this situation; that is, suppose we want to find a formula giving the quantity, Q, in terms of time, t. Making a mathematical model often involves looking at a graph and deciding what kind of function has that shape. The graph in Figure 1.21 looks like an exponential decay function, upside down. What actually decays is the difference between the saturation level, S, and the quantity, Q, in the blood. Suppose the difference between the saturation level and the quantity in the body is given by the formula

$$\text{Difference} = (\text{Initial difference}) \cdot (0.3)^t$$

with t in hours. Since the difference is $S - Q$, and the initial value of this difference is $S - 0 = S$:

$$S - Q = S \cdot (0.3)^t.$$

Solving for Q as a function of t gives

$$Q = S - S \cdot (0.3)^t$$
$$Q = f(t) = S \cdot \left(1 - (0.3)^t\right).$$

Notice that the graph of this function is an upside-down exponential. As t gets larger, $(0.3)^t$ gets smaller, so Q gets closer to S. Using "$\rightarrow$" to mean "tends to," we can say $(0.3)^t \rightarrow 0$ as $t \rightarrow \infty$. This shows that

$$Q = S(1 - (0.3)^t) \rightarrow S(1 - 0) = S \quad \text{as} \quad t \rightarrow \infty$$

confirming that the graph of $Q = S(1 - (0.3)^t)$ has a horizontal asymptote at $Q = S$.

The Family of Exponential Functions

The formula $P = P_0 a^t$ gives a family of exponential functions with parameters P_0 (the initial quantity) and a (the base, or growth factor). The base is as important for an exponential function as the slope is for a linear function. We assume $a > 0$ and $a \neq 1$. Then the base tells you whether the function is increasing ($a > 1$) or decreasing ($0 < a < 1$). Since a is the factor by which P changes when t is increased by 1, large values of a mean fast growth; values of a near 0 mean fast decay. (See Figures 1.22 and 1.23.)

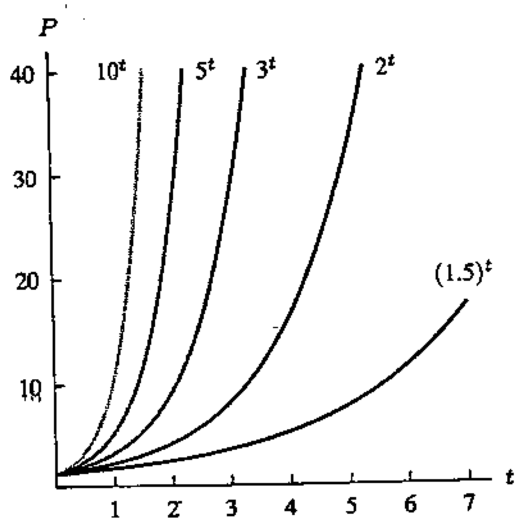


Figure 1.22: Exponential growth: $P = a^t$, $a > 1$

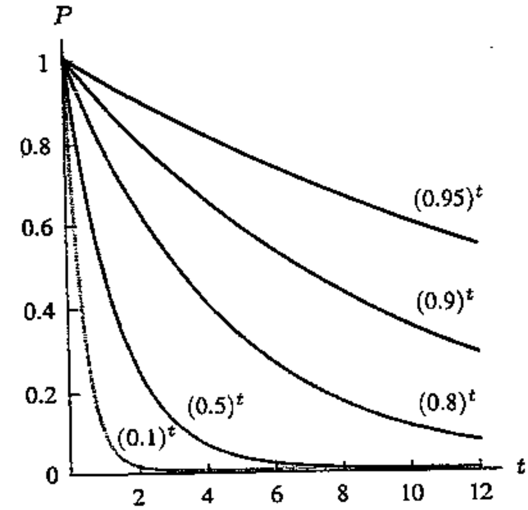


Figure 1.23: Exponential decay: $P = a^t$, $0 < a < 1$

Alternative Formula for the Exponential Function

Exponential growth is often described in terms of percentages. For example, the population of Mexico is growing at 2.6% per year; in other words, the growth factor is $a = 1.026$. Similarly, each foot of clay removes 20% of the pollution from jet fuel, so the decay factor is $a = 1 - 0.20 = 0.8$. In general, the following formulas apply.

If r is the *growth* rate, then $a = 1 + r$, and

$$P = P_0 a^t = P_0(1+r)^t.$$

If r is the *decay* rate, then $a = 1 - r$, and

$$P = P_0 a^t = P_0(1-r)^t.$$

Note, for example, that $r = 0.05$ when the percentage growth rate is 5%.

Example 2 Suppose that $Q = f(t)$ is an exponential function of t. If $f(20.1) = 88.2$ and $f(20.3) = 91.4$:
(a) Find the base. (b) Find the percentage growth rate. (c) Evaluate $f(21.4)$.

Solution (a) Let

$$Q = Q_0 a^t.$$

Then

$$88.2 = Q_0 a^{20.1} \quad \text{and} \quad 91.4 = Q_0 a^{20.3}.$$

Dividing gives

$$\frac{91.4}{88.2} = \frac{Q_0 a^{20.3}}{Q_0 a^{20.1}} = a^{0.2}.$$

Solve for the base, a:

$$a = \left(\frac{91.4}{88.2}\right)^{1/0.2} = 1.195.$$

(b) Since $a = 1.195$, the growth rate is $r = 0.195 = 19.5\%$.
(c) We want to find $f(21.4) = Q_0 a^{21.4} = Q_0(1.195)^{21.4}$. First let's find Q_0 from the equation $91.4 = Q_0(1.195)^{20.3}$. Solving gives $Q_0 = 2.457$. Thus,

$$f(21.4) = 2.457(1.195)^{21.4} = 111.19.$$

Definition and Properties of Exponents

Below we list the definitions and rules that are used to manipulate exponents.

Definition of Zero, Negative, and Fractional Exponents

$$a^0 = 1, \quad a^{-1} = \frac{1}{a}, \quad \text{and, in general, } a^{-x} = \frac{1}{a^x}$$

$$a^{1/2} = \sqrt{a}, \quad a^{1/3} = \sqrt[3]{a}, \quad \text{and, in general, } a^{1/n} = \sqrt[n]{a}.$$

Rules for Computing Using Exponents

1. $a^x \cdot a^t = a^{x+t}$ For example, $2^4 \cdot 2^3 = (2\cdot2\cdot2\cdot2)\cdot(2\cdot2\cdot2) = 2^7$.
2. $\dfrac{a^x}{a^t} = a^{x-t}$ For example, $\dfrac{2^4}{2^3} = \dfrac{2\cdot2\cdot2\cdot2}{2\cdot2\cdot2} = 2^1$.
3. $(a^x)^t = a^{xt}$ For example, $(2^3)^2 = 2^3 \cdot 2^3 = 2^6$.

Problems for Section 1.3

1. The number of cancer cells grows slowly at first but then grows with increasing rapidity. Draw a possible graph of the number of cancer cells against time.
2. Each year the world's annual consumption of electricity rises. In addition, each year the increase in annual consumption also rises. Sketch a possible graph of the annual world consumption of electricity as a function of time.
3. A drug is injected into a patient's bloodstream over a five-minute interval. During this time, the quantity in the blood increases linearly. After five minutes the injection is discontinued, and the quantity then decays exponentially. Sketch a graph of the quantity versus time.

4. When there are no other steroid hormones (for example, estrogen) in a cell, the rate at which steroid hormones diffuse into the cell is fast. The rate slows down as the amount in the cell builds up. Sketch a possible graph of the quantity of steroid hormone in the cell against time, assuming that initially there are no steroid hormones in the cell.

5. Each of the functions in Table 1.10 is increasing, but each increases in a different way. Which of the graphs in Figure 1.24 below best fits each function?

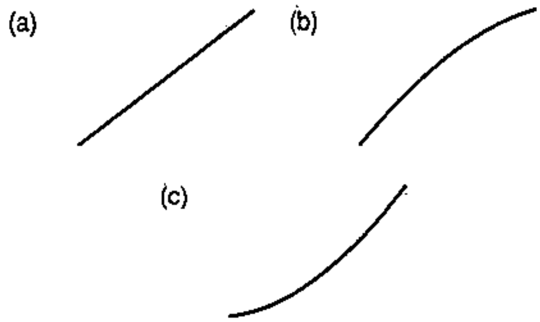


Figure 1.24

TABLE 1.10

t	$g(t)$	$h(t)$	$k(t)$
1	23	10	2.2
2	24	20	2.5
3	26	29	2.8
4	29	37	3.1
5	33	44	3.4
6	38	50	3.7

6. Each of the functions in Table 1.11 decreases, but each decreases in a different way. Whic of the graphs in Figure 1.25 below best fits each function?

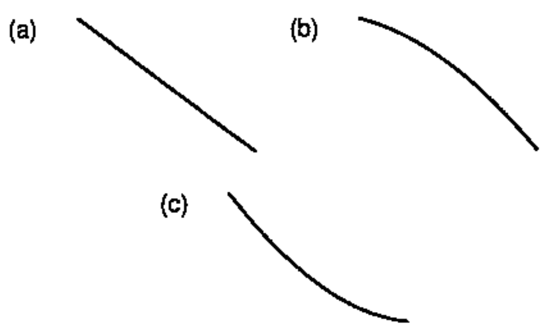


Figure 1.25

TABLE 1.11

x	$f(x)$	$g(x)$	$h(x)$
1	100	22.0	9.3
2	90	21.4	9.1
3	81	20.8	8.8
4	73	20.2	8.4
5	66	19.6	7.9
6	60	19.0	7.3

7. Match up the function values in Table 1.12 with the formulas

$$y = a(1.1)^s\ ,\ \ y = b(1.05)^s\ ,\ \ y = c(1.03)^s,$$

assuming a, b, and c are constants. Note that the function values have been rounded to two decimal places.

TABLE 1.12

s	$h(s)$	s	$f(s)$	s	$g(s)$
2	1.06	1	2.20	3	3.47
3	1.09	2	2.42	4	3.65
4	1.13	3	2.66	5	3.83
5	1.16	4	2.93	6	4.02
6	1.19	5	3.22	7	4.22

For Problems 8–9, find a possible formula for the functions represented by the data.

8.

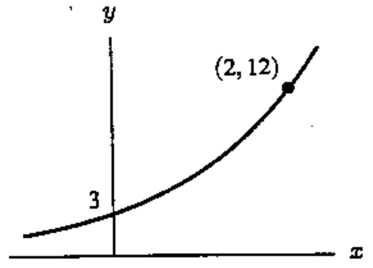
x	0	1	2	3
$f(x)$	4.30	6.02	8.43	11.80

9.

t	0	1	2	3
$g(t)$	5.50	4.40	3.52	2.8[illegible]

Find possible equations for the graphs in Problems 10–13.

10.



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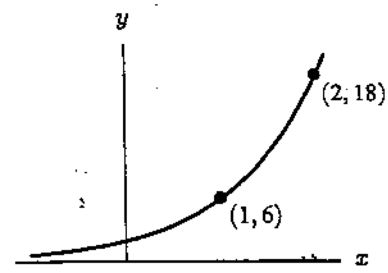

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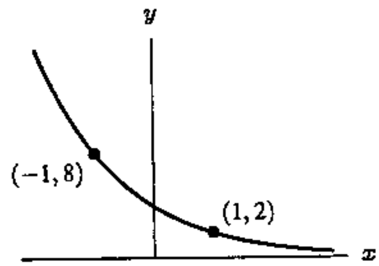

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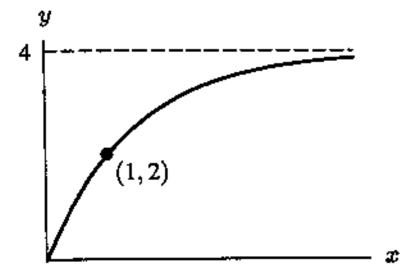

14. (a) The half-life of radium-226 is 1620 years. Write a formula for the quantity, Q, of radium left after t years, if the initial quantity is Q_0.
(b) What percentage of an original amount of radium is left after 500 years?

15. In the early 1960s, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and got into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 1990?

16. When the Olympic Games were held outside Mexico City in 1968, there was much discussion about the effect the high altitude (7340 feet) would have on the athletes. Assuming air pressure decays exponentially at 0.4% every 100 feet, by what percentage is air pressure reduced by moving from sea level to Mexico City?

17. A population is known to be growing exponentially. Estimate the doubling time of the population shown by the graph in Figure 1.26, and verify graphically that the doubling time is independent of where you start on the graph.

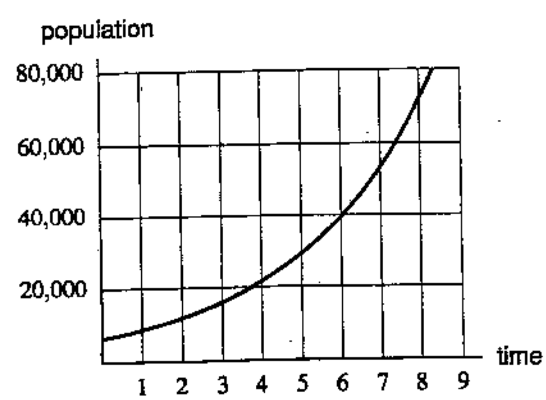

Figure 1.26

18. A certain region has a population of 10,000,000 and an annual growth rate of 2%. Estimate the doubling time by trial and error.

19. A certain radioactive substance decays exponentially in such a way that after 10 years, 70% of the initial amount remains. Find an expression for the quantity remaining after any number of years t. How much will be present after 50 years? What is the half-life? How long will it be before only 20% of the original amount is left? Before only 10% is left? (Use trial and error where necessary.)

20. Assume that the median price, P, of a home rose from \$50,000 in 1970 to \$100,000 in 1990. Let t be the number of years since 1970.
    (a) Assume the increase in housing prices has been linear. Find an equation for the line representing price, P, in terms of t. Use this equation to complete column (a) of Table 1.13. Work with the price in units of \$1000.
    (b) If instead the housing prices have been rising exponentially, determine an equation of the form $P = P_0a^t$ which would represent the change in housing prices from 1970–1990, and complete column (b) of Table 1.13.
    (c) On the same set of axes, sketch the functions represented in column (a) and column (b) of Table 1.13.

TABLE 1.13

t	(a) Linear growth price in \$1000 units	(b) Exponential growth price in \$1000 units
0	50	50
10		
20	100	100
30		
40		

Countries with very high inflation rates often publish monthly rather than yearly inflation figures, because monthly figures are less alarming. Problems 21–22 involve such high rates, which are called *hyperinflation*.

21. In 1989, US inflation was 4.6% a year. In 1989 Argentina had an inflation rate of about 33% a month.
    (a) What is the yearly equivalent of Argentina's 33% monthly rate?
    (b) What is the monthly equivalent of the US 4.6% yearly rate?

22. Between December 1988 and December 1989, Brazil's inflation rate was 1290% a year. (This means that between 1988 and 1989, prices increased by a factor of $1 + 12.90 = 13.90$.)
    (a) What would an article which cost 1000 cruzeiros (the Brazilian currency unit) in 1988 cost in 1989?
    (b) What was Brazil's monthly inflation rate during this period?

1.4 POWER FUNCTIONS

Power functions are an important family of functions. A *power function* is one in which the dependent variable is proportional to a power of the independent variable. For example, the area, A, of a square of side s is given by

$$A = f(s) = s^2.$$

The volume, V, of a sphere of radius r is

$$V = f(r) = \frac{4}{3}\pi r^3.$$

Both of these are *power functions*. So is the function which describes how the gravitational attraction of the earth varies with distance. If g is the force of gravitational attraction on a unit mass at a distance r from the earth, Newton's Inverse Square Law of Gravitation says that

$$g = \frac{k}{r^2} \quad \text{or} \quad g = kr^{-2}$$

where k is a positive constant.

In general, a power function has the form

$$y = f(x) = kx^p$$

where k and p are any constants.

In this section we will compare various power functions with one another and with the exponential functions. Since $y = f(x) = mx$ (with m a constant) is also a power function (because $x = x^1$, the first power), linear functions are included in the comparison too.

Positive Integral Powers: $y = x, y = x^2, y = x^3, \ldots$

First, we'll look at functions of the form $f(x) = x^n$, with n a positive integer. Figures 1.27 and 1.28 show that the graphs of these functions fall into two groups: the odd powers and the even powers. All the odd powers (x, x^3, x^5, and so on) are increasing everywhere and their graphs are symmetric about the origin. All odd powers above $n = 1$ have a bend, or "seat," at the origin. The even powers, on the other hand, are first decreasing and then increasing, making them $\bigcup$-shaped with symmetry about the y-axis. The even powers are concave up everywhere, whereas the odd ones (greater than 1) are concave down for negative x and concave up for positive x. All odd and even powers, however, go through the points $(0, 0)$ and $(1, 1)$.

Figure 1.29 shows that the higher the power of x, the faster the function climbs. For large values of x (in fact, for all $x > 1$), $y = x^5$ is above $y = x^4$, which is above $y = x^3$, and so on. Not only are the higher powers larger, but they are *much* larger. This is because if $x = 100$, for example, 100^5 is one hundred times as big as 100^4 which is one hundred times as big as 100^3. As x gets larger (written as $x \to \infty$), any positive power of x completely swamps all lower powers of x. We say that, as $x \to \infty$, higher powers of x *dominate* lower powers.

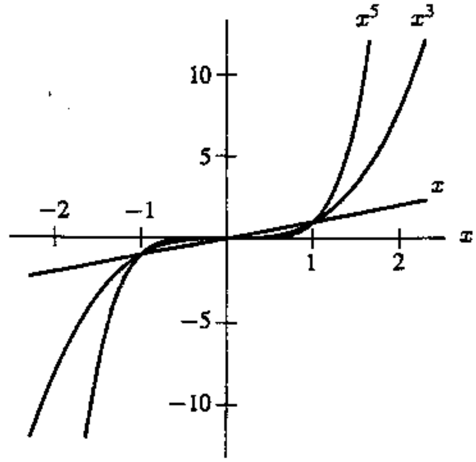

Figure 1.27: Odd powers of x

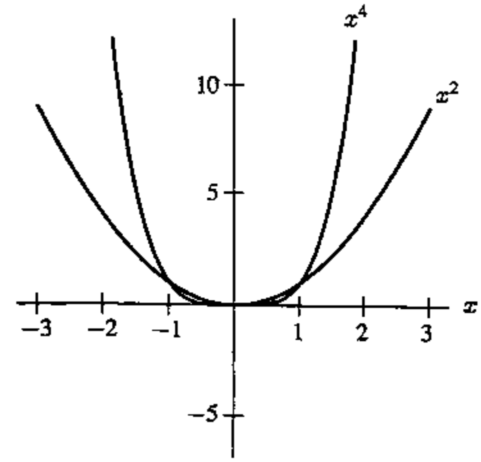

Figure 1.28: Even powers of x

The close-up view near the origin in Figure 1.30 shows an entirely different story. For x between 0 and 1, the order is reversed: x^3 is bigger than x^4, which is bigger than x^5. (Try $x = 0.1$ to confirm this.) The fact that higher powers of x climb faster is true for large values of x but not for small. For big values of x, the highest powers are largest; for values of x near zero, smaller powers dominate.

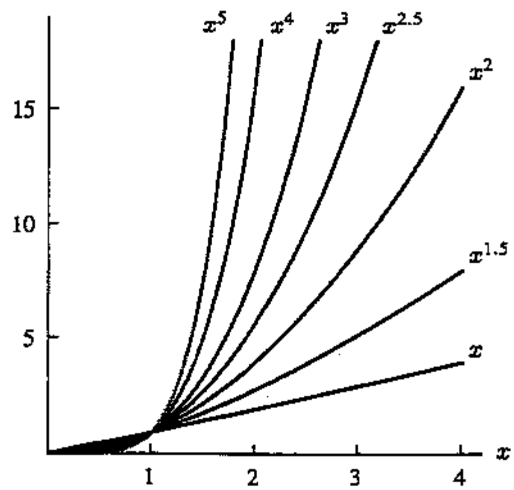

Figure 1.29: Powers of x: Which is largest for large values of x?

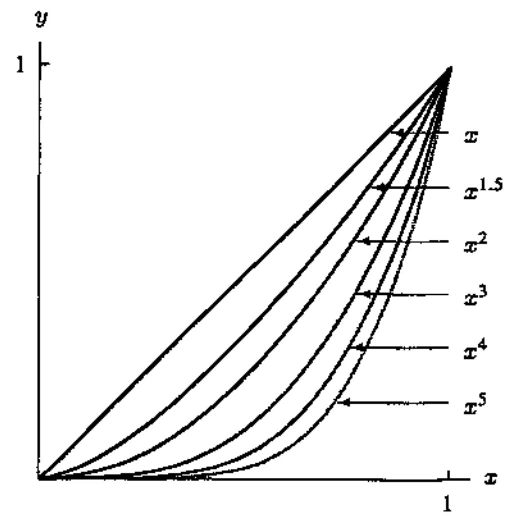

Figure 1.30: Between 0 and 1: Small powers of x dominate

Zero and Negative Integral Powers: $y = x^0, y = x^{-1}, y = x^{-2}, \ldots$

The function $y = x^0 = 1$ has a graph that is a horizontal line. Rewriting

$$y = x^{-1} = \frac{1}{x} \quad \text{and} \quad y = x^{-2} = \frac{1}{x^2}$$

makes it easier to see that as x increases, the denominators increase and the functions decrease. The graphs of $y = x^{-1}$ and $y = x^{-2}$ have both the x and y axes as asymptotes. (See Figure 1.31.) For $x > 1$, the graph of $y = x^{-2}$ is below that of $y = x^{-1}$, and both must stay below $y = x^0 = 1$.

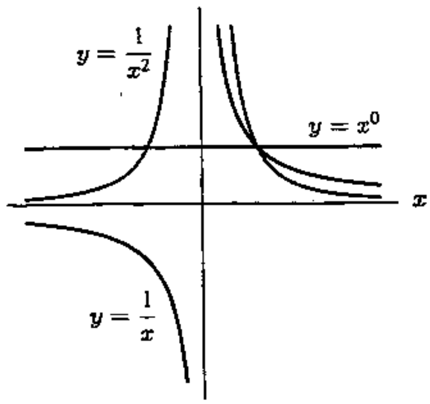

Figure 1.31: Comparison of zero and negative powers of x

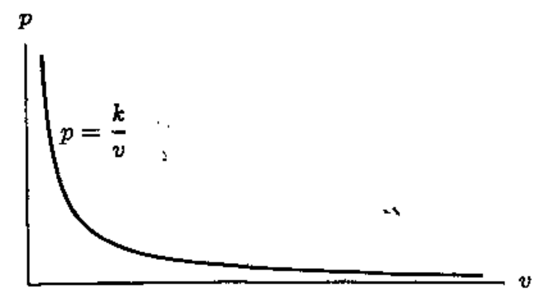

Figure 1.32: Graph of pressure, p, against volume, v, for Boyle's law

Example 1 Plot a graph of pressure against volume for a fixed quantity of gas at a constant temperature. Use the fact that pressure is inversely proportional to volume.

Solution Think of a fixed quantity of air — for example, inside a cylinder in a car engine. If the volume of the air is decreased (by moving the pistons), the pressure of the air will be increased. Conversely, if the volume is increased, the pressure will decrease. For an ideal gas, Boyle's law gives the exact relationship between pressure and volume, provided the temperature is constant. It says pressure times volume is constant, or

$$pv = k$$

with k positive. So

$$p = \frac{k}{v} = kv^{-1}.$$

The relationship $p = k/v$ is equivalent to saying p is inversely proportional to v. When v is large, p is small, and when v is small, p is large. For any (positive) value of k, the graph has the shape shown in Figure 1.32. Both axes are asymptotes, showing that as the volume tends to infinity, the pressure tends to zero, and vice versa. The shape is known as a rectangular *hyperbola*. The power function $p = k/v = kv^{-1}$ differs from exponential decay in that it is undefined for $v = 0$, so this graph does not cross the vertical axis. In addition, it approaches the horizontal axis more slowly than the exponential function.

Example 2 Quantum mechanics predicts that the force between two gas molecules has two components: an attractive force which is approximately proportional to r^{-7} (where r is the distance between the molecules) and a repulsive force approximately proportional to r^{-13}. How does the net force vary with r?

Solution A repulsive force is usually considered to be positive, whereas an attractive force is negative. Thus, we write $F = -ar^{-7} + br^{-13}$, where a, b are positive constants. If r is very small, r^{-13} is larger than r^{-7}, so $F \approx br^{-13}$ and the net force is repulsive. (This occurs when the molecules are so close together that the protons in the nuclei repel one another.) For larger r, r^{-7} is larger than r^{-13}, and $F \approx -ar^{-7}$, making the net force attractive. As $r \to \infty$, all the forces die away to zero.

Positive Fractional Powers: $y = x^{1/2}, y = x^{1/3}, y = x^{3/2}, \ldots$

The function giving the side of a square, s, in terms of its area, A, involves a root, or fractional power:

$$s = \sqrt{A} = A^{1/2}.$$

Similarly, the equation relating the average number of species found on an island and the size of the island involves a fractional power. If N is the number of species and A is the area of the island, observations have shown[4] that approximately

$$N = k\sqrt[3]{A} = kA^{1/3}$$

where k is a constant depending on the region of the world in which the island is found.

We will now look at functions of the form $y = x^{m/n} = \sqrt[n]{x^m}$. Since some fractional powers such as $x^{1/2}$ involve roots and are defined only for positive x and 0, we frequently restrict the domain of positive fractional powers of x to $x \geq 0$. Many calculators will not allow you to raise a negative number to a fractional power.

Figure 1.33 shows that for large x (in fact, all $x > 1$), the graph of $y = x^{1/2}$ is below the graph of $y = x$, and $y = x^{1/3}$ is below $y = x^{1/2}$. This is reasonable since, for example, $10^{1/2} = \sqrt{10} \approx 3.16$ and $10^{1/3} = \sqrt[3]{10} \approx 2.15$, so $10^{1/3} < 10^{1/2} < 10$. Between $x = 0$ and $x = 1$, the situation is reversed, and $y = x^{1/3}$ is on top. (Why?) Not surprisingly, $y = x^{3/2}$ is between $y = x$ and $y = x^2$ for all x.

The other important feature to notice about the graphs of $y = x^{1/2}$ and $y = x^{1/3}$ is that they bend in a direction opposite to that of the graphs of $y = x^2$ and x^3. For example, the graph of $y = x^2$ is climbing faster and faster as x increases; it is concave up. On the other hand, the graphs of $y = x^{1/2}$ and $y = x^{1/3}$ are climbing slower and slower; they are concave down. Despite this, all these functions do become infinitely large as x increases.

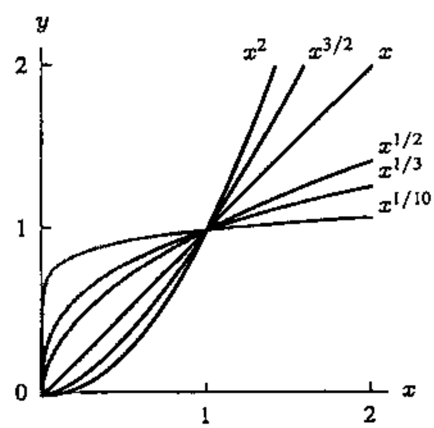

Figure 1.33: Comparison of some fractional powers of x

What Effect Do Coefficients Have?

We know that $x^2 < x^3$ for all $x > 1$. But which is larger, $50x^2$ or x^3? Eventually, $50x^2 < x^3$ too. In fact, $50x^2 < x^3$ for all $x > 50$. (See Figure 1.34.) The graphs of $y = x^2$ and $y = x^3$ cross at $x = 1$, whereas graphs of $y = 50x^2$ and $y = x^3$ cross at $x = 50$. Thus, the effect of the factor of 50 is to change the point at which the graphs cross. However, x^3 ends up on top in both cases: provided the coefficients are positive, as $x \to \infty$, the higher power is always larger eventually.

[4] *Scientific American*, September 1989, p. 112.

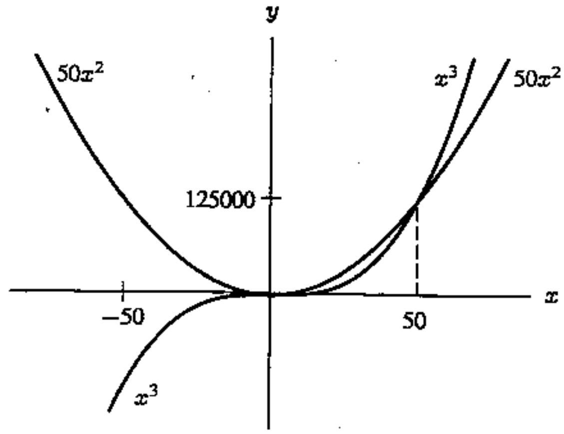

Figure 1.34: Graph of $y = x^3$ lies above graph of $y = 50x^2$ for large positive x

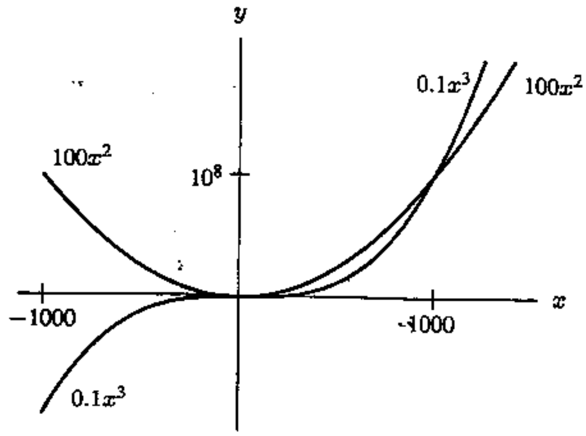

Figure 1.35: For large positive x, $y = 0.1x^3$ dominates $y = 100x^2$

Example 3 Which of $y = 100x^2$ and $y = 0.1x^3$ is larger as $x \to \infty$?

Solution Since $x \to \infty$, we are looking at large positive values of x, where the higher power will eventually be larger, or dominate. Thus, $y = 0.1x^3$ will be larger. (See Figure 1.35.)

Example 4 Sketch a global view of $f(x) = -x^3$, $g(x) = 40x^4$, and $h(x) = -0.1x^5$. Which function has the largest positive values as $x \to \infty$? Which function has the largest positive values as $x \to -\infty$ (i.e., as x gets more and more negative)?

Solution As $x \to \infty$, $g(x) = 40x^4$ is the only function which is positive. As $x \to -\infty$, the graph of $h(x) = -0.1x^5$ is eventually (for $x < -400$) above the graph of the other functions, so $h(x)$ has the largest positive values as $x \to -\infty$ (See Figure 1.36.) Notice that, for large x, the values of $f(x) = -x^3$ are so much smaller in magnitude than the values of the other functions that the graph of $f(x)$ cannot be seen in the far-away view.

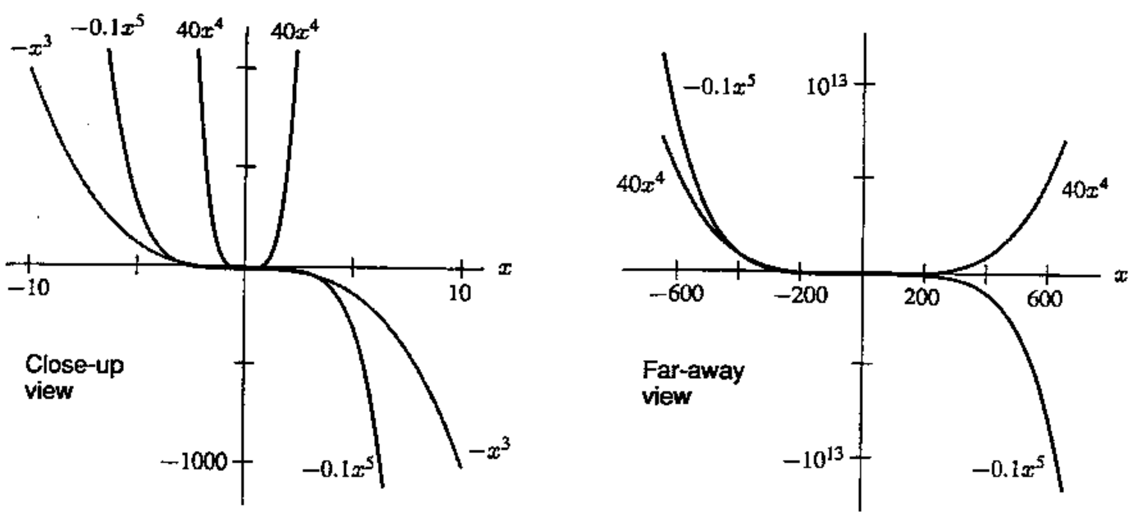

Figure 1.36: As $x \to \infty$, $g(x) = 40x^4$ dominates; as $x \to -\infty$, $h(x) = -0.1x^5$ dominates

Exponentials and Power Functions: Which Dominate?

In everyday language, *exponential* is often used to imply very fast growth. But do exponential functions always grow faster than power functions?

Let's consider $y = 2^x$ and $y = x^3$. The close-up, or local, view in Figure 1.37(a) shows that between $x = 2$ and $x = 5$, the graph of $y = 2^x$ lies below the graph of $y = x^3$. But the more global, or far–away, view in Figure 1.37(b) shows that the exponential function $y = 2^x$ eventually overtakes $y = x^3$. And Figure 1.37(c), which gives a very far–away, or global view, shows that, for large x, x^3 is insignificant compared to 2^x. Indeed, 2^x is growing so much faster than x^3 that its graph appears almost vertical—in comparison to the more leisurely climb of x^3.

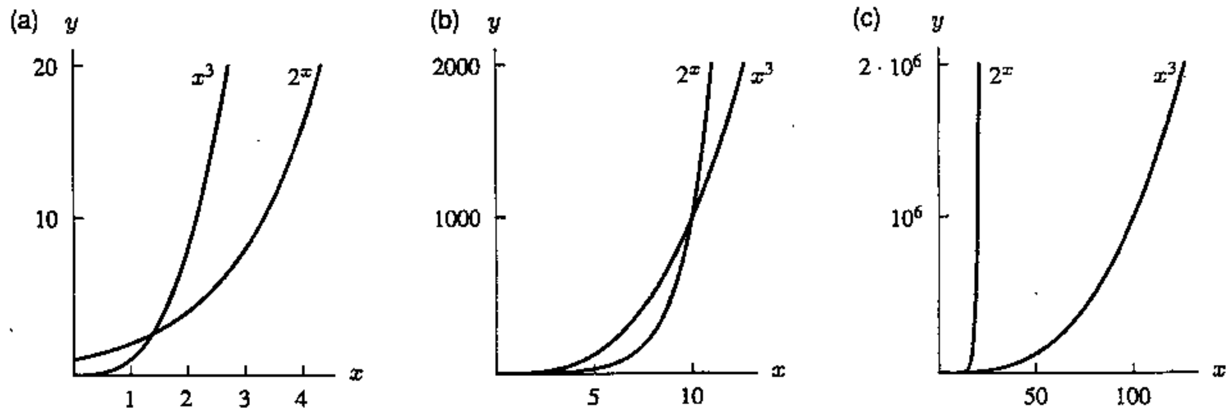

Figure 1.37: Comparison of $y = 2^x$ and $y = x^3$: $y = 2^x$ eventually dominates $y = x^3$

In fact, *every* exponential growth function eventually dominates *every* power function. Although an exponential function may be below a power function for some values of x, if you look at large enough x values, a^x (with $a > 1$) will eventually dominate x^n, no matter what n is. Two more examples are presented in Figure 1.38 and Table 1.14.

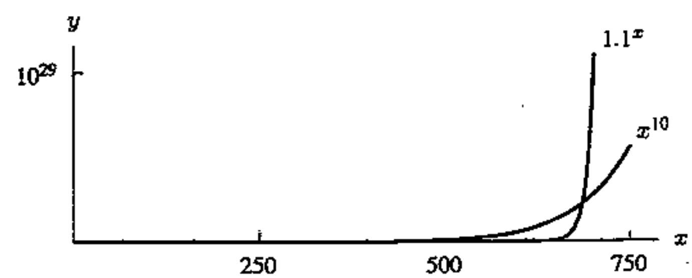

Figure 1.38: Exponential function eventually dominates power function

TABLE 1.14 *Comparison of x^{100} and 1.01^x*

x	x^{100}	1.01^x
10^4	10^{400}	$1.6 \cdot 10^{43}$
10^5	10^{500}	$1.4 \cdot 10^{432}$
10^6	10^{600}	$2.4 \cdot 10^{4321}$

Can you guess what happens in the case of negative powers and negative exponents? For example, consider $y = 2^{-x}$ and $y = x^{-2}$. Since $y = 2^{-x} = 1/2^x$ and $y = x^{-2} = 1/x^2$, knowing that 2^x is eventually larger than x^2 tells you that 2^{-x} is eventually smaller than x^{-2}. Hence $y = 2^{-x}$ is eventually below $y = x^{-2}$. (See Figure 1.39.) This behavior is also typical: Every exponential decay function will eventually approach 0 faster than every power function with a negative exponent.

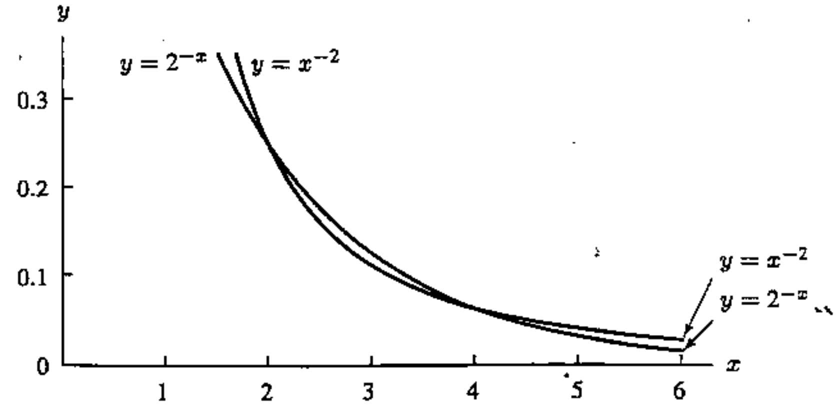

Figure 1.39: Comparison of $y = 2^{-x}$ and $y = x^{-2}$: Exponential dies away faster

Problems for Section 1.4

1. Simplify each of the following: (a) $8^{2/3}$ (b) $9^{-3/2}$
2. Sketch graphs of $y = x^{1/2}$ and $y = x^{2/3}$ on the same axes. Which function has larger values as $x \to \infty$?
3. What happens to the value of $y = x^4$ as $x \to \infty$? As $x \to -\infty$?
4. What happens to the value of $y = -x^7$ as $x \to \infty$? As $x \to -\infty$?
5. Sketch a graph of $y = x^{-4}$.
6. On a graphing calculator or computer, plot graphs of the following functions, first for $-5 \le x \le 5$, $-100 \le y \le 100$, and then for $-1.2 \le x \le 1.2$, $-2 \le y \le 2$.
   (a) $y = x, y = x^3, y = x^6, y = x^9$
   (b) $y = x, y = x^4, y = x^7, y = x^{10}$
   Observe the general shape of these functions: Do the odd powers have the same general shape? What about the even powers? Which function is largest in magnitude for big x? For x near 0? Is this what you expected?
7. Do some calculations using specific values of x to verify that $y = x^{1/3}$ is above $y = x^{1/2}$ and that $y = x^{1/2}$ is above $y = x$ for $0 < x < 1$.
8. (a) Use a graphing calculator (or a computer) to plot the graphs of x^3, x^4, and x^5 on the interval $-0.1 \le x \le 0.1$. Determine an appropriate range for y so that all powers will be distinguishable in the viewing rectangle.
   (b) Plot the same graphs for $-100 \le x \le 100$, and determine an appropriate range for y.
9. By hand, sketch global pictures of $f(x) = x^3$ and $g(x) = 20x^2$ on the same axes. Which function has larger values as $x \to \infty$?
10. By hand, sketch pictures of $f(x) = x^5$, $g(x) = -x^3$, and $h(x) = 5x^2$ on the same axes. Which has larger positive values as $x \to \infty$? As $x \to -\infty$?
11. By trial and error, use a calculator to find to two decimal places the point near $x = 10$ at which $y = 2^x$ and $y = x^3$ cross.
12. Use a graphing calculator to find the point(s) of intersection of the graphs of $y = (1.06)^x$ and $y = 1 + x$.
13. For what values of x is $4^x > x^4$?

14. For what values of x is $3^x > x^3$? (Note: You will need to think about how to deal with the fact that the graphs of 3^x and x^3 are relatively close together for values of x near 3.)

15. According to the April 1991 issue of *Car and Driver*, an Alfa Romeo going at 70 mph requires 177 feet to stop. Assuming that the stopping distance is proportional to the square of velocity, find the stopping distances required by an Alfa Romeo going at 35 mph and at 140 mph (its top speed).

16. Poiseuille's law gives the rate of flow, R, of a gas through a cylindrical pipe in terms of the radius of the pipe, r, for a fixed drop in pressure. Assume a constant drop in pressure throughout the remainder of this problem.
    (a) Determine a formula for Poiseuille's Law, given that the rate of flow is proportional to the fourth power of the radius.
    (b) If $R = 400$ cm^3/sec in a pipe of radius 3 cm for a certain gas, determine an explicit formula for the rate of flow of that gas through a pipe of radius r cm.
    (c) What is the rate of flow of the same gas through a pipe with a 5-cm radius?

17. The values of three functions are given in Table 1.15. One function is of the form $y = ab^t$, one is of the form $y = at^2$, and one is of the form $y = bt^3$. Which function is which?

TABLE 1.15

t	$f(t)$	t	$g(t)$	t	$h(t)$
2.0	4.40	1.0	3.00	0.0	2.04
2.2	5.32	1.2	5.18	1.0	3.06
2.4	6.34	1.4	8.23	2.0	4.59
2.6	7.44	1.6	12.29	3.0	6.89
2.8	8.62	1.8	17.50	4.0	10.33
3.0	9.90	2.0	24.00	5.0	15.49

18. Values of three functions are contained in Table 1.16. (The numbers have been rounded to two decimal places.) Two are power functions and one is an exponential. One of the power functions is a quadratic and one a cubic. Which one is exponential? Which one is quadratic? Which one is cubic?

TABLE 1.16

x	$f(x)$	x	$g(x)$	x	$k(x)$
8.4	5.93	5.0	3.12	0.6	3.24
9.0	7.29	5.5	3.74	1.0	9.01
9.6	8.85	6.0	4.49	1.4	17.66
10.2	10.61	6.5	5.39	1.8	29.19
10.8	12.60	7.0	6.47	2.2	43.61
11.4	14.82	7.5	7.76	2.6	60.91

19. Owing to improved seed types and new agricultural techniques, the grain production of a region has been increasing. Over a 20-year period, annual production (in millions of tons) was as follows:

1970	1975	1980	1985	1990
5.35	5.90	6.49	7.05	7.64

At the same time the population (in millions) was:

1970	1975	1980	1985	1990
53.2	56.9	60.9	65.2	69.7

(a) Find a linear or exponential function which approximately fits each set of data. (Pick whichever type of function fits better.)
(b) If this region was self-supporting in this grain in 1970, was it self-supporting between 1970 and 1990? (Being self-supporting means that each person has enough of the grain. How does the amount of grain each person has in later years compare?)
(c) What are your predictions for the future if the trends continue?

20. Use a graphing calculator (or a computer) to graph $y = x^4$ and $y = 3^x$. Determine the appropriate domains and ranges that will give each of the graphs in Figure 1.40.

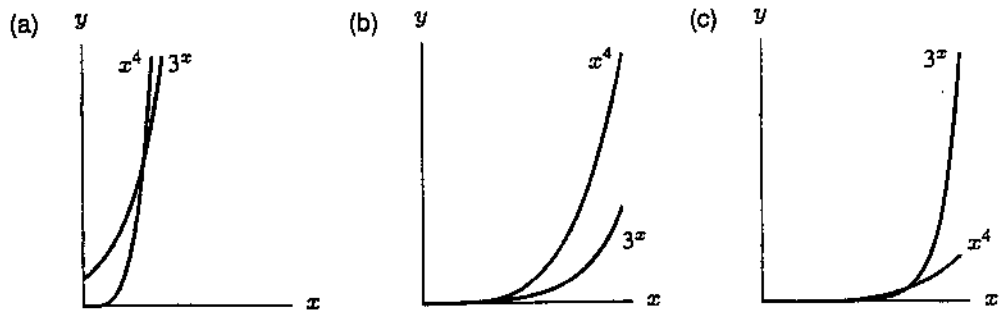

Figure 1.40

1.5 INVERSE FUNCTIONS

From Distance to Time and Back

On August 18, 1989, Arturo Barrios of Mexico set a world record in the 10,000-meter run with a time of 27 minutes and 8.23 seconds. His times, in seconds, at 2000-meter intervals are recorded in Table 1.17, where $f(d)$ is the number of seconds Barrios took to complete the first d meters of the race. For example, Barrios ran the first 4000 meters in 650.1 seconds, so $f(4000) = 650.1$. The function f is useful to athletes planning to compete with Barrios.

Let us now change our point of view and ask for distances rather than times. If we ask how far Barrios ran during the first 650.1 seconds of his race, the answer is clearly 4000 meters. Going backwards in this way from numbers of seconds to numbers of meters gives a function called the *inverse function* of f, denoted by f^{-1}. Thus, $f^{-1}(t)$ is the number of meters that Barrios ran during the first t seconds of his race. To find values of f^{-1}, we can either read Table 1.17 backwards, or use Table 1.18 which contains values of f^{-1}.

The two functions f and f^{-1} convey the same information, but they express it differently. For example, the fact that Barrios ran the first 6000 meters in 975.5 seconds can be written with either f or f^{-1}:

$$f(6000) = 975.5 \quad \text{or} \quad f^{-1}(975.5) = 6000.$$

TABLE 1.17 *Barrios's running time*

d (meters)	$f(d)$ (seconds)
0	0.00
2000	325.90
4000	650.10
6000	975.50
8000	1307.00
10000	1628.23

TABLE 1.18 *Distance run by Barrios*

t (seconds)	$f^{-1}(t)$ (meters)
0.00	0
325.90	2000
650.10	4000
975.50	6000
1307.00	8000
1628.23	10000

The independent variable for f is the dependent variable for f^{-1}, and vice versa. The domains and ranges of f and f^{-1} are also interchanged. The domain of f is all distances d such that $0 \le d \le 10000$, which is the range of f^{-1}. The range of f is all times t, such that $0 \le t \le 1628.23$, which is the domain of f^{-1}.

Definition of an Inverse Function

Not every function f has an inverse, a problem we will discuss later in this section. But when an inverse exists, it is defined as follows:

$$f^{-1}(t) = d \quad \text{means} \quad f(d) = t$$

The notation f^{-1} to denote an inverse function is perhaps unfortunate, as it is easy to confuse with a reciprocal, which the inverse function is not. However, there's no changing such well-established notation!

Example 1 The temperature at which water boils decreases as altitude increases, a fact important to cooks. Let $f(h)$ be the boiling point (°C) of water at an altitude of h meters above sea level during standard atmospheric conditions. What is the meaning in practical terms of $f^{-1}(90)$ and of $f^{-1}(90) = 3000$? Evaluate $f^{-1}(100)$.

Solution The function f goes from altitude to temperature, so f^{-1} goes back from temperature to altitude. Thus, $f^{-1}(90)$ is the altitude in meters at which the boiling point of water is 90°C. The equation $f^{-1}(90) = 3000$ means that the boiling point of water is 90°C at an altitude of 3000 meters. The equation $f(3000) = 90$ has the same meaning. Since the boiling point of water is 100°C at sea level (where the altitude is 0 meters), we must have $f^{-1}(100) = 0$.

Formulas for Inverse Functions

If a function is defined by a formula, it is sometimes possible to find a formula for the inverse function as well. In Section 1.1, we looked at the snow tree cricket, whose chirp rate, C, in chirps per minute, is approximated by the formula

$$C = f(T) = 4T - 160$$

where T is the temperature in degrees Fahrenheit. So far we have used this formula to predict the chirp rate from the temperature. But it is perfectly possible to use this formula backwards to calculate the temperature from the chirp rate.

Example 2 Find the formula for the function giving temperature in terms of the number of cricket chirps per minute; that is, find the inverse function f^{-1} such that

$$T = f^{-1}(C).$$

Solution We know $C = 4T - 160$. We solve for T, giving

$$T = \frac{C}{4} + 40$$

so

$$f^{-1}(C) = \frac{C}{4} + 40.$$

When Does a Function Have an Inverse?

If a function has an inverse, we say it is *invertible*. Not all function have inverses, and the best way to understand which ones do have inverses is to look at an example which does not.

Consider the flight of the Mercury spacecraft *Freedom 7*, which carried the first American, Alan Shepard, Jr., into space in May 1961. The spacecraft rose to an altitude of 116 miles, and then came down into the sea for a total flight time of 15 minutes. The function $f(t)$ giving the altitude in miles t minutes after lift-off does not have an inverse. To see why not, try to decide on a value for $f^{-1}(100)$. Clearly, $f^{-1}(100)$ should tell us the time when the altitude of the spacecraft was 100 miles. However, there are two such times, one when the spacecraft was ascending and one when it was descending. (See Figure 1.41.)

Since functions must take single definite values, there is no inverse function f^{-1}. The reason why the altitude function $f(t)$ does not have an inverse is that the altitude first increases and then decreases, so each altitude corresponds to two times, t. The reason why the Barrios time and chirp rate functions do have inverses is that they both increase everywhere. Thus, each running time, t, corresponds to some unique distance, d, and any biologically reasonable number of chirps, C, corresponds to some particular temperature, T.

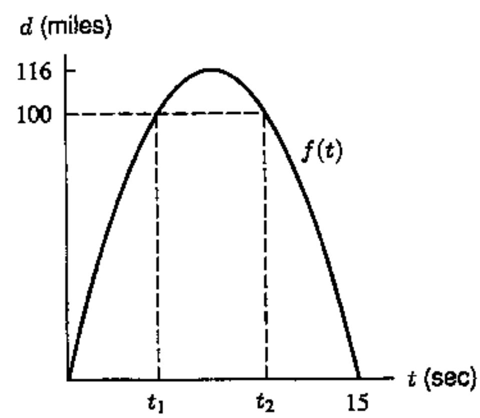

Figure 1.41: Two times, t_1 and t_2, at which altitude of spacecraft is 100 miles

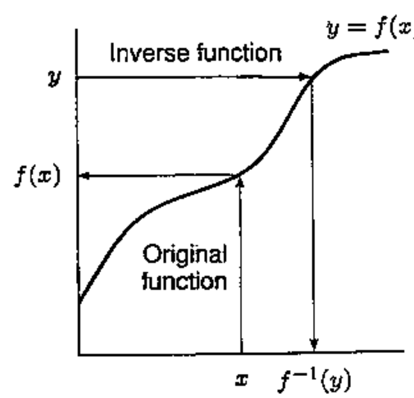

Figure 1.42: Why an increasing function has an inverse

A function does not have to be increasing everywhere to have an inverse, but the graph in Figure 1.42 does suggest when an inverse will exist. The original function, f, takes you from x to y, as shown in Figure 1.42. Since having an inverse means there is a function going from y to x, the crucial question is whether you can get back. In other words, does each y value correspond to a unique x value? If so, there's an inverse; if not, there isn't. This principle may be stated geometrically, as follows:

> A function has an inverse if (and only if) its graph intersects any horizontal line at most once.

Thus, for example, the function $f(x) = x^2$ does not have an inverse because many horizontal lines intersect the parabola twice. (See Figure 1.43.)

Graphs of Inverse Functions

The graph of $f(x) = x^3$ shows that this function is increasing everywhere and so has an inverse. To find the inverse, we solve

$$y = x^3$$

for x, giving

$$x = \sqrt[3]{y}.$$

Thus, the inverse function is

$$f^{-1}(y) = \sqrt[3]{y}$$

or, if we want to call the variable x,

$$f^{-1}(x) = \sqrt[3]{x}.$$

The graphs of $y = x^3$ and $y = x^{1/3}$ are shown in Figure 1.44. Notice that these graphs are the reflections of one another about the line $y = x$. For example, $(8, 2)$ is on the graph of $y = x^{1/3}$ because $2 = 8^{1/3}$, and $(2, 8)$ is on the graph of $y = x^3$ because $8 = 2^3$. The points $(8, 2)$ and $(2, 8)$ are reflections of one another about the line $y = x$. In general:

> The graph of f^{-1} is the reflection of the graph of f about the line $y = x$.

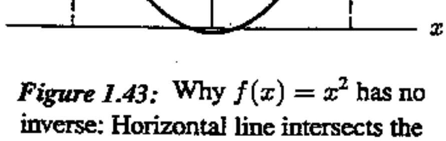

Figure 1.43: Why $f(x) = x^2$ has no inverse: Horizontal line intersects the graph twice

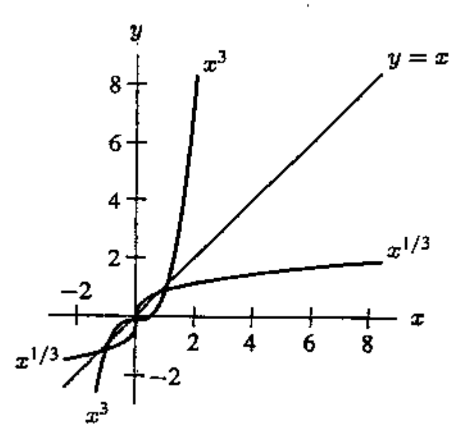

Figure 1.44: Graphs of inverse functions, $y = x^3$ and $y = x^{1/3}$, are reflections in the line $y = x$

Problems for Section 1.5

1. Let $f(x)$ equal the temperature (°F) when the column of mercury in a particular thermometer is x inches long. What is the meaning of $f^{-1}(75)$ in practical terms?

For Problems 2–6, decide whether the function f is invertible.

2. $f(d)$ is the total number of gallons of fuel an airplane has used by the end of d minutes of a particular flight.
3. $f(t)$ equals the number of customers in Macy's department store at t minutes past noon on December 18, 1993.
4. $f(x)$ is the volume in liters of x kilograms of water at 4°C.
5. $f(w)$ is the cost in cents of mailing a letter that weighs w grams.
6. $f(n)$ is the number of students in your calculus class whose birthday is on the n^{th} day of the year.
7. Write a table of values for f^{-1}, where f is as given below. The domain of f is the integers from 1 to 7. State the domain of f^{-1}.

x	1	2	3	4	5	6	7
$f(x)$	3	−7	19	4	178	2	1

8. The function $f(x) = x^3 + x + 1$ is invertible. Use a graphing calculator to give an approximate value for $f^{-1}(20)$.

For Problems 9–11, use a graphing calculator or computer to sketch the graphs of the following functions, and decide whether or not they are invertible.

9. $f(x) = x^2 + 3x + 2$
10. $f(x) = x^3 - 5x + 10$
11. $f(x) = x^3 + 5x + 10$

12. The cost of producing q articles is given by the function

$$C = f(q) = 100 + 2q.$$

   (a) Find a formula for the inverse function.
   (b) Explain in practical terms what the inverse function tells you.

13. A kilogram weighs about 2.2 pounds.
   (a) Write a formula for the function, f, which gives an object's mass in kilograms, k, as a function of its weight in pounds, p.
   (b) Find a formula for the inverse function of f. What does this inverse function tell you, in practical terms?
14. Suppose f is invertible and increasing. What can you say about whether its inverse is increasing or decreasing?
15. If a function f is invertible and concave up, what can you say about the concavity of its inverse?

16. Figure 1.45 is the graph[5] of the function f, where $f(t)$ is the number (in millions) of motor vehicles registered in the world in the year t. (In 1988, one-third of the registered vehicles in the world were in the United States.)
   (a) Is f invertible? Explain.
   (b) What is the meaning of $f^{-1}(400)$ in practical terms? Evaluate $f^{-1}(400)$.
   (c) Sketch the graph of f^{-1}.

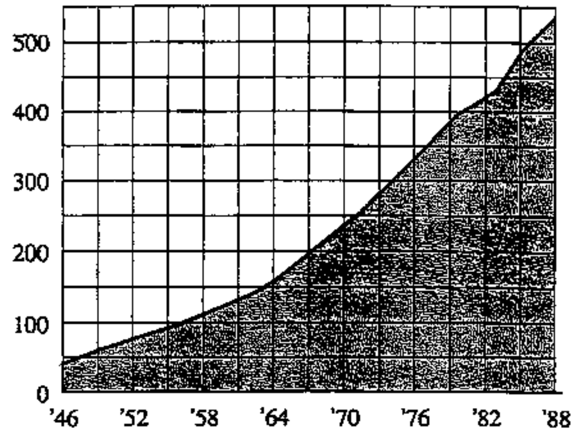

Figure 1.45: World population of motor vehicles

1.6 LOGARITHMS

In Section 1.3, we set up a function approximating the population of Mexico (in millions) as

$$P = f(t) = 67.38(1.026)^t$$

where t is the number of years since 1980. Writing the function this way shows that we are thinking of the population as a function of time, and that we believe the population to be 67.38 million in 1980 and to grow by 2.6% every year.

Now suppose that instead of calculating the population, we want to find when the population is expected to reach 100 million. This means we want to find the value of t for which

$$100 = f(t) = 67.38(1.026)^t.$$

Since the exponential function is always increasing and is eventually more than 100, there's exactly one value of t making $P = 100$. How should we find it? A reasonable way to start is trial and error. Taking $t = 10$ and $t = 20$ we get

$$P = f(10) = 67.38(1.026)^{10} = 87.1\ldots \quad (\text{so } t = 10 \text{ is too small})$$
$$P = f(20) = 67.38(1.026)^{20} = 112.58\ldots \quad (\text{so } t = 20 \text{ is too large})$$

Some more experimenting leads to

$$P = f(15) = 67.38(1.026)^{15} \approx 99.0$$
$$P = f(16) = 67.38(1.026)^{16} \approx 101.6$$

[5] From D. Blerics and P. Walzer, "Energy for Motor Vehicles," *Scientific American*, September 1990.